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# ON THE CRITICAL ONE COMPONENT REGULARITY FOR 3-D NAVIER-STOKES SYSTEM

BY JEAN-YVES CHEMIN AND PING ZHANG

ABSTRACT. – Given an initial data $v_0$ with vorticity $\Omega_0 = \nabla \times v_0$ in $L^{\frac{3}{2}}$ (which implies that $v_0$ belongs to the Sobolev space $H^{\frac{1}{2}}$), we prove that the solution $v$ given by the classical Fujita-Kato theorem blows up in a finite time $T^\star$ only if, for any $p$ in $]4,6[$ and any unit vector $e$ in $\mathbb{R}^3$, there holds $\int_0^{T^\star} \|v(t)\cdot e\|^p_{H^{\frac{1}{2}+\frac{2}{p}}}\,dt = \infty$. We remark that all these quantities are scaling invariant under the scaling transformation of Navier-Stokes system.

RÉSUMÉ. – On considère une donnée initiale $v_0$ dont la vorticité $\Omega_0 = \nabla \times v_0$ appartient à $L^{\frac{3}{2}}$ (ce qui implique que $v_0$ appartient à l'espace de Sobolev $H^{\frac{1}{2}}$). Nous démontrons que si la solution $v$ de l'équation de Navier-Stokes tridimensionnelle associée à $v_0$ par le théorème de Fujita-Kato développe une singularité à l'instant $T^\star$ (fini) alors, pour tout $p$ dans l'intervalle $]4,6[$ et tout vecteur unitaire $e$ de $\mathbb{R}^3$, on a $\int_0^{T^\star} \|v(t)\cdot e\|^p_{H^{\frac{1}{2}+\frac{2}{p}}}\,dt = \infty$. Remarquons que toutes ses quantités sont invariantes par les changements d'échelle de l'équation de Navier-Stokes.

## 1. Introduction

In the present work, we investigate necessary conditions for the breakdown of the regularity of regular solutions to the following 3-D homogeneous incompressible Navier-Stokes system

$$(NS) \qquad \begin{cases} \partial_t v + \operatorname{div}(v\otimes v) - \Delta v + \nabla\Pi = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0, \end{cases}$$

where $v = (v^1, v^2, v^3)$ stands for the velocity of the fluid and $\Pi$ for the pressure. Let us first recall some fundamental results proved by J. Leray in his seminal paper [19].

THEOREM 1.1. – *Let us consider an initial data $v_0$ which belongs to the inhomogeneous Sobolev space $H^1_{\rm in}(\mathbb{R}^3)$. There exists a (unique) maximal positive time of existence $T^\star$ such that a unique solution $v$ of $(NS)$ exists on $[0,T^\star[\times\mathbb{R}^3$, which is continuous with value in $H^1_{\rm in}(\mathbb{R}^3)$ and the gradient of which belongs to $L^2_{\rm loc}([0,T^\star[;H^1_{\rm in}(\mathbb{R}^3))$. Moreover, if $\|v_0\|_{L^2}\|\nabla v_0\|_{L^2}$ is small enough, then $T^\star$ is infinite. If $T^\star$ is finite, we have, for any $q$ greater than* 3,

$$\forall\, t<T^\star\,,\ \|v(t)\|_{L^q}\geq \frac{C_q}{(T^\star-t)^{\frac12\left(1-\frac3q\right)}}\,\cdot$$

Let us also mention that in [19], J. Leray proved also the existence (but not the uniqueness) of global weak (turbulent in J. Leray's terminology) solutions of $(NS)$ with initial data only in $L^2(\mathbb{R}^3)$. In the present paper, we only deal with solutions which are regular to be unique.

In [19], J. Leray emphasized two basic facts about the homogeneous incompressible Navier-Stokes system: the $L^2$ energy estimate and the scaling invariance.

Because the vector field $v$ is divergence free, the energy estimate formally reads

$$\frac12\frac{d}{dt}\|v(t)\|^2_{L^2}+\|\nabla v(t)\|^2_{L^2}=0.$$

After time integration, this gives

$$\frac12\|v(t)\|^2_{L^2}+\int_0^t\|\nabla v(t')\|^2_{L^2}dt'=\frac12\|v_0\|^2_{L^2}. \tag{1.1}$$

This estimate is the cornerstone of the proof of the existence of global turbulent solution to $(NS)$ done by J. Leray in [19]. The energy estimate relies (formally) on the fact that if $v$ is a divergence free vector field, $(v\cdot\nabla f|f)_{L^2}=0$ and that $(\nabla p|v)_{L^2}=0$. In the present work, we shall use the more general fact that for any divergence free vector field $v$ and any function $a$, we have

$$\int_{\mathbb{R}^3}v(x)\cdot\nabla a(x)|a(x)|^{p-2}a(x)\,dx=0\qquad\text{for any }\ p\in\,]1,\infty[.$$

This will lead to the $L^p$ type energy estimate.

The scaling invariance is the fact that if $v$ is a solution of $(NS)$ on $[0,T]\times\mathbb{R}^3$ associated with an initial data $v_0$, then $\lambda v(\lambda^2t,\lambda x)$ is also a solution of $(NS)$ on $[0,\lambda^{-2}T]\times\mathbb{R}^3$ associated with the initial data $\lambda v_0(\lambda x)$. The importance of this point can be illustrated by this sentence coming from [19] *"... les équations aux dimensions permettent de prévoir* a priori *presque toutes les inégalités que nous écrirons ... "*(1) The scaling property is also the foundation of the Kato theory which gives a general method to solve (locally or globally) the incompressible Navier-Stokes equation in critical spaces, i.e., spaces whose norms are invariant under the scaling. In the present work, we only use such scaling invariant spaces. Let us exhibit some examples of scaling invariant norms. For $p\geq 2$, the norms of

$$L^p_t(H^{\frac12+\frac2p})\quad\text{and}\quad L^p_t(L_x^{3+\frac{6}{p-2}})$$

are scaling invariant norms. The spaces $H^{\frac12}$ and $L^3$ are scaling invariant spaces for the initial data $v_0$. Let us point out that in the case when the space dimension is two, the energy norm which appears in Relation (1.1) is scaling invariant. This allows to prove that in the two dimensional case, turbulent solutions are unique and regular.

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(1) This can be translated by *"The scaling allows to guess almost all the inequalities written in this paper".*

The first result of local (and global for small initial data) wellposedness of $(NS)$ in a scaling invariant space was proved by H. Fujita and T. Kato in 1964 (see [13]) for initial data in the homogeneous Sobolev space $H^{\frac{1}{2}}$. More precisely, we have the following statement.

THEOREM 1.2. – *Let us consider an initial data $v_0$ in the homogeneous Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$. There exists a (unique) maximal positive time of existence $T^\star$ such that a unique solution $v$ of $(NS)$ exists on $[0,T^\star[\times\mathbb{R}^3$ which is continuous in time with value in $H^{\frac{1}{2}}(\mathbb{R}^3)$ and belongs to $L^2_{\rm loc}([0,T^\star[;H^{\frac{3}{2}}(\mathbb{R}^3))$. Moreover, if the quantity $\|v_0\|_{H^{\frac{1}{2}}}$ is small enough, then $T^\star$ is infinite. If $T^\star$ is finite, we have, for any $q$ greater than* 3,

$$\forall\, t<T^\star\,,\ \|v(t)\|_{L^q}\geq C_q\frac{1}{(T^\star-t)^{\frac{1}{2}(1-\frac{3}{q})}}\,\cdot$$

Let us point out that the above necessary condition for blow-up implies that

$$T^\star<\infty\Longrightarrow\int_0^{T^\star}\|v(t)\|^p_{L^q}dt=\infty\quad\text{with}\quad\frac{2}{p}+\frac{3}{q}=1\quad\text{and}\quad p<\infty. \tag{1.2}$$

Let us mention that it is possible to prove this theorem without using the energy estimate and this theorem is true for a large class of systems which have the same scaling as the incompressible Navier-Stokes system.

Using results related to the energy estimate, L. Iskauriaza, G. A. Serëgin and V. Sverak proved in 2003 the end point case of (1.2) when $p$ is infinite (see [16]). This remarkable result has been extended to Besov space with negative index (see [10]). Let us also mention a blow-up criterion proposed by Beirão da Veiga [3], which states that if the maximal time $T^\star$ of existence of a regular solution $v$ to $(NS)$ is finite, then we have

$$\int_0^{T^\star}\|\nabla v(t)\|^p_{L^q}dt=\infty\quad\text{with}\quad\frac{2}{p}+\frac{3}{q}=2\quad\text{for}\quad q\geq\frac{3}{2}\,\cdot \tag{1.3}$$

Let us observe that because of the fact that homogeneous bounded Fourier multipliers maps $L^p$ into $L^p$, this criteria is equivalent, for $q$ is finite, to

$$\int_0^{T^\star}\|\Omega(t)\|^p_{L^q}dt=\infty\quad\text{where}\quad\Omega\stackrel{\text{def}}{=}\nabla\times v. \tag{1.4}$$

In this case when $q$ is infinite, this criteria is the classical Beale-Kato-Majda theorem (see [2]) which is in fact a result about Euler equation and where the viscosity plays no role.

In the present paper, we want to establish necessary conditions for breakdown of regularity of solutions to $(NS)$ given by Theorem 1.2 in term of the scaling invariant norms of one component of the velocity field. Because we shall use the $L^{\frac{3}{2}}$ norm of the vorticity, we work with solutions given by the following theorem, which are a little bit more regular than that given by Theorem 1.2.

THEOREM 1.3. – *Let us consider an initial data $v_0$ with vorticity $\Omega_0=\nabla\times v_0$ in $L^{\frac{3}{2}}$. Then a unique maximal solution $v$ of $(NS)$ exists in the space $C([0,T^*[;H^{\frac{1}{2}})\cap L^2_{\rm loc}([0,T^\star[;H^{\frac{3}{2}})$ for some positive time $T^*$, and the vorticity $\Omega=\nabla\times v$ is continuous on $[0,T^*[$ with value in $L^{\frac{3}{2}}$ and $\Omega$ satisfies*

$$|\nabla\Omega|\,|\Omega|^{-\frac{1}{4}}\in L^2_{\rm loc}([0,T^\star[;L^2).$$