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LONG TIME DYNAMICS FOR DAMPED KLEIN-GORDON EQUATIONS

BY Nicolas BURQ, Geneviève RAUGEL AND Wilhelm SCHLAG

ABSTRACT. – For general nonlinear Klein-Gordon equations with dissipation we show that any finite energy radial solution either blows up in finite time or asymptotically approaches a stationary solution in $H^1 \times L^2$. In particular, any global in positive times solution is bounded in positive times. The result applies to standard energy subcritical focusing nonlinearities $|u|^{p-1}u$, $1 < p < (d+2)/(d-2)$ as well as to any energy subcritical nonlinearity obeying a sign condition of the Ambrosetti-Rabinowitz type. The argument involves both techniques from nonlinear dispersive PDEs and dynamical systems (invariant manifold theory in Banach spaces and convergence theorems).

RÉSUMÉ. – Nous démontrons que toute solution radiale d'énergie finie d'une classe générale d'équations de Klein-Gordon amorties ou bien explose en temps positif fini ou bien converge en temps positif vers une solution stationnaire dans $H^1 \times L^2$. En particulier, toute solution globale en temps positif est bornée en temps positif. Ce résultat s'applique aux non-linéarités focalisantes, sous-critiques pour l'énergie, $|u|^{p-1}u$, $1 < p < (d+2)/(d-2)$, comme à toute non-linéarité, sous-critique pour l'énergie, remplissant une condition de signe de type Ambrosetti-Rabinowitz. La preuve fait appel, à la fois, à des techniques propres aux équations non linéaires dispersives et à des arguments de systèmes dynamiques (variétés invariantes dans des espaces de Banach et théorèmes de convergence).

1. Introduction

Nonlinear dispersive evolution equations such as the wave and Schrödinger equations have been investigated for decades. For defocusing power-type energy subcritical or critical nonlinearities the theory is developed, while the energy supercritical powers are wide open. For semilinear focusing equations the picture is less complete for long-term dynamics. These

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equations exhibit finite-time blowup, small data global existence and scattering, as well as time-independent solutions (solitons). For the energy critical wave equation

$$\Box u = u^5, \quad (t,x) \in \mathbb{R}^{1+3},$$
$$(u(0), \partial_t u(0)) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

in the radial setting, Duyckaerts, Kenig, and Merle [16] achieved a breakthrough by showing that all global trajectories can be described as a superposition of a finite number of rescalings of the ground state $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$ plus a radiation term which is asymptotic to a free wave. This work introduces the novel *exterior energy* estimates.

The subcritical case appears to require different techniques, however. The focusing subcritical Klein-Gordon equation in $\mathbb{R}^d$, $1 \le d \le 6$ (for the case $d \ge 7$, see [7]), takes the form

$$\begin{aligned} &\partial_t^2 u - \Delta u + u - |u|^{\theta-1} u = 0, \\ &(u(0), \partial_t u(0)) = (\varphi_0, \varphi_1) \in \mathcal{H}, \end{aligned} \tag{1.1}$$

where $\mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $\alpha \ge 0$ and

$$1 < \theta < \theta^*, \text{ with } \theta^* = \frac{d+2}{d-2}. \tag{1.2}$$

We will limit our study to the case of radial functions

$$\mathcal{H}_{\mathrm{rad}} = H^1_{\mathrm{rad}}(\mathbb{R}^d) \times L^2_{\mathrm{rad}}(\mathbb{R}^d).$$

The energy functional E^θ below plays an important role in the analysis of the behavior of the solutions of (1.1). This energy functional is given by

$$E^\theta(\varphi_0, \varphi_1) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla \varphi_0|^2 + \frac{1}{2}\varphi_0^2 + \frac{1}{2}\varphi_1^2 - \frac{1}{\theta+1} |\varphi_0|^{\theta+1} \right) dx. \tag{1.3}$$

For the Klein-Gordon Equation (1.1), it is known (see [35], [3], [14], [29] and [10] for example) that (1.1) admits a unique positive radial stationary solution $(Q_g, 0)$ (the ground state solution), which minimizes the energy $E^\theta(.,0)$ in the class of all nonzero stationary solutions $(Q,0)$ in $\mathcal{H}$, that is,

$$0 < E^\theta(Q_g, 0) = \min\{E(Q,0) \mid Q \in H^1(\mathbb{R}^d), Q \neq 0, -\Delta Q + Q - |Q|^{\theta-1} Q = 0\}$$

The behavior of solutions of (1.1) with initial data $(\varphi_0, \varphi_1) \in \mathcal{H}$ with energy $E^\theta(\varphi_0, \varphi_1) < E^\theta(Q_g, 0)$ is rather well understood since these solutions remain in the so-called Payne-Sattinger sets (see [32]) for all positive times. In these Payne-Sattinger domains, the solutions either blow-up in finite time or globally exist and scatter to 0 (for a description of this phenomenon, we refer for example to the book [30]).

Nakanishi and the third author [30] described the asymptotics of solutions provided the energy $E^\theta(\varphi_0, \varphi_1)$ is only slightly larger than the ground state energy. They showed the following trichotomy in forward time of (i) blowup in finite time (ii) global existence and scattering to zero (iii) global existence and scattering to the ground state. They formulated this trichotomy in terms of the center-stable manifold associated with the ground state $(Q_g, 0)$.

It is also well-known that this equation has an infinite number of radial equilibrium points $(e_\ell, 0)$ with a prescribed number $\ell \geq 1$ of zeros (these are called *nodal solutions*, see for example [4]). Unfortunately, one knows almost nothing about the uniqueness and the hyperbolicity of those nodal solutions (In [15] the authors obtain uniqueness results for nodal solutions but for sub-linear nonlinearities). This lack of information prevents the description of the behavior of the solutions $\vec{u}(t)$ of (1.1) whose initial data (φ_0, φ_1) have an energy $E^\theta(\varphi_0, \varphi_1)$ much larger than the one of the ground state $(Q_g, 0)$.

In 1985 Cazenave [9] established the following dichotomy: solutions of (1.1) either blow up in finite time or are global and bounded in $\mathscr{H}$, provided $1 < \theta < +\infty$, if $d = 1, 2$ with $\theta \leq 5$ if $d = 2$ and $1 < \theta \leq \frac{d}{d-2}$ if $d \geq 3$.

In view of these previous results, a natural conjecture is that any *global, radial, finite energy* solution of (1.1) should scatter toward an equilibrium. However, this result seems to be presently out of reach of the usual approaches. A more accessible model is the focusing subcritical *damped* Klein-Gordon equation

$$\begin{aligned} &\partial_t^2 u - \Delta u + u + 2\alpha\partial_t u - |u|^{\theta-1}u = 0, \\ &(u(0), \partial_t u(0)) = (\varphi_0, \varphi_1) \in \mathscr{H}. \end{aligned} \tag{1.4}$$

In 1998 Feireisl [18], for the dissipative case $\alpha > 0$, gave an independent proof of the boundedness of the global solutions of (1.4), when $d \geq 3$ and $1 < \theta < 1 + \min(\frac{2}{d-2}, \frac{4}{d})$ (for the case $d = 1$, see his earlier paper [17]). On the other hand, the results of Cazenave should extend to the damped case. However, the proofs of Cazenave [9] and of Feireisl [18] do not seem to extend to nonlinearities satisfying $\frac{d}{d-2} < \theta < \frac{d+2}{d-2}$, when $d \geq 3$, where one needs to use Strichartz estimates in the various a priori estimates rather than Gagliardo-Nirenberg-Sobolev inequalities.

Another motivation for studying the *damped* equation is that, by playing on the damping term and considering the damping $2\alpha(t,x)\partial_t u$ or even the nonlinear damping $2\alpha|\partial_t u|^{\delta-1}\partial_t u$, one should be able to exhibit much richer behaviors (from the dynamics point of view). In this paper, we develop a robust approach to the problem of long-term asymptotics of the general *radial* energy subcritical Klein-Gordon equations with (arbitrarily small) dissipation. Our main result is the following dichotomy.

THEOREM 1.1. – *Let $\alpha > 0$ and $d \leq 6$. Then,*

1. *either the solutions of* (1.4) *in $\mathscr{H}_{\rm rad}$ blow up in finite positive time,*
2. *or they are global in positive time and converge to an equilibrium point.*

In particular, all global in positive time solutions are bounded for positive time.

We notice that this theorem is a particular case of Theorem 1.2 below. In [7], we will partly generalize this dichotomy to non-radial solutions.

Actually the above dichotomy holds for some more general nonlinearities and, in this paper, we consider the damped Klein-Gordon equation in $\mathbb{R}^d$, $d \leq 6$ (for the case $d \geq 7$, see [7]),

$$\begin{aligned} &\partial_t^2 u + 2\alpha\partial_t u - \Delta u + u - f(u) = 0, \\ &(u(0), \partial_t u(0)) = (\varphi_0, \varphi_1) \in \mathscr{H}_{\rm rad}, \end{aligned} \tag{$(KG)_\alpha$}$$