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PARTIAL HASSE INVARIANTS ON SPLITTING MODELS OF HILBERT MODULAR VARIETIES

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ABSTRACT. – Let F be a totally real field of degree g, and let p be a prime number. We construct g partial Hasse invariants on the characteristic p fiber of the Pappas-Rapoport splitting model of the Hilbert modular variety for F with level prime to p, extending the usual partial Hasse invariants defined over the Rapoport locus. In particular, when p ramifies in F, we solve the problem of lack of partial Hasse invariants. Using the stratification induced by these generalized partial Hasse invariants on the splitting model, we prove in complete generality the existence of Galois pseudo-representations attached to Hecke eigenclasses of paritious weight occurring in the coherent cohomology of Hilbert modular varieties mod p^m, extending a previous result of M. Emerton and the authors which required p to be unramified in F.

RÉSUMÉ. – Soient F un corps totalement réel de degré g et p un nombre premier. On construit g invariants de Hasse partiels sur la fibre de caractéristique p du modèle scindé de Pappas-Rapoport de la variété modulaire de Hilbert pour F de niveau premier à p. Ils généralisent les invariants de Hasse partiels usuels sur le lieu de Rapoport. En particulier, nous résolvons le problème du manque des invariants de Hasse lorsque p est ramifié dans F. Utilisant la stratification sur le modèle scindé induite par ces invariants de Hasse, nous prouvons en toute généralité l'existence de pseudo-représentations galoisiennes attachées aux systèmes de Hecke qui apparaissent dans la cohomolgie cohérente des variétés modulaires modulo p^n. Ceci étend un résultat des auteurs avec E. Emerton où on a supposé que p est non-ramifié dans F.

1. Introduction

Let F be a totally real field of degree $g > 1$ over $\mathbb{Q}$, and let p be a prime number. Fix a large enough finite extension $\mathbb{F}$ of $\mathbb{F}_p$. The (characteristic p fiber of the) *Deligne-Pappas moduli space* $\mathcal{M}_{\mathbb{F}}^{\mathrm{DP}}$ parameterizes g-dimensional abelian schemes A/S defined over an $\mathbb{F}$-scheme S and endowed with an action of the ring of integers of F, a polarization, and a suitable

prime-to-p level structure (cf. [4]). The scheme $\mathcal{M}_{\mathbb{F}}^{\mathrm{DP}}$ is normal, and its smooth locus $\mathcal{M}_{\mathbb{F}}^{\mathrm{Ra}}$, called the *Rapoport locus*, parameterizes those abelian schemes A/S whose sheaf of invariant differentials is locally free of rank one as an $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S)$-module (cf. [16]).

In [8] and [1], F. Andreatta and E. Goren construct some modular forms defined over the Rapoport locus, called the *partial Hasse invariants*, which factor the determinant of the Hasse-Witt matrix of the universal abelian scheme $\mathcal{A}_{\mathbb{F}}^{\mathrm{Ra}}$ with respect to the action of the totally real field F.

When p is unramified in F there are exactly g partial Hasse invariants, which give rise to a good stratification of the Hilbert moduli scheme $\mathcal{M}_{\mathbb{F}}^{\mathrm{Ra}} = \mathcal{M}_{\mathbb{F}}^{\mathrm{DP}}$ (cf. [9] and [1]). On the other hand, when p ramifies in F the Rapoport locus is open and dense in $\mathcal{M}_{\mathbb{F}}^{\mathrm{DP}}$ with complement of codimension two, and the partial Hasse invariants of [1] do not extend to the Deligne-Pappas moduli space. In addition, the number of such operators is strictly less than g. For example, when p is totally ramified in F, only one partial Hasse invariant is defined in [1] on $\mathcal{M}_{\mathbb{F}}^{\mathrm{Ra}}$: it is a gth root of the determinant of the Hasse-Witt matrix, up to sign. The lack of partial Hasse invariants when p ramifies in F was in particular an obstruction in extending to the ramified settings the results proved in the unramified case by M. Emerton and the authors in [7].

To remedy this, we work in this paper with the (characteristic p fiber of the) *splitting model* of the Hilbert modular scheme constructed by G. Pappas and M. Rapoport in [15], and made explicit by S. Sasaki in [17]. This is a smooth scheme $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$ over $\mathbb{F}$ endowed with a birational morphism $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}} \to \mathcal{M}_{\mathbb{F}}^{\mathrm{DP}}$ which is an isomorphism if and only if p is unramified in F, and which induces an isomorphism from a suitable open dense subscheme of $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$ onto $\mathcal{M}_{\mathbb{F}}^{\mathrm{Ra}}$. There is a natural notion of automorphic line bundles on $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$, and hence of Hilbert modular forms.

We construct g modular forms of non-parallel weight defined over the entire Pappas-Rapoport splitting model $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$, and extending the classical Hasse invariants over the Rapoport locus. Moreover, when p ramifies in F some of the operators we construct do not have a classical counterpart (see below).

We briefly discuss some of the ideas of this construction. Let us assume for simplicity that $p\mathcal{O}_F = \mathfrak{p}^e$ for some integer $e \geq 1$, and that the inertial degree of p in F is equal to f. Let $\varpi_{\mathfrak{p}}$ denote a choice of uniformizer for the completed local ring $\mathcal{O}_{F_{\mathfrak{p}}}$, and denote by $\mathbb{F}$ the residue field of F at $\mathfrak{p}$. Let $\tau_1, \ldots, \tau_f : \mathbb{F} \to \overline{\mathbb{F}}$ denote the embeddings of $\mathbb{F}$ into its algebraic closure, ordered so that $\sigma \circ \tau_i = \tau_{i+1}$, where i stands for i (mod f), and σ denotes the arithmetic Frobenius. For an abelian scheme A/S defined over an $\mathbb{F}$-scheme S and endowed with real multiplication by $\mathcal{O}_F$, we denote by $\omega_{A/S,j}$ the direct summand of the sheaf of invariant differentials of A/S on which $W(\mathbb{F}) \subset \mathcal{O}_{F_{\mathfrak{p}}}$ acts through τ_j. The sheaf $\omega_{A/S,j}$ is a locally free $\mathcal{O}_S$-module of rank e.

The Pappas-Rapoport splitting model $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$ parameterizes isomorphism classes of tuples $(A, \lambda, i, \underline{\mathscr{F}} = (\mathscr{F}_j^{(l)})_{j,l})/S$ where A is a Hilbert-Blumenthal abelian scheme defined over an $\mathbb{F}$-scheme S, endowed with a polarization λ and a prime-to-p level structure i, and for each $j = 1, \ldots, f$ we are given a filtration of $\omega_{A/S,j}$:

$$0 = \mathscr{F}_j^{(0)} \subset \mathscr{F}_j^{(1)} \subset \cdots \subset \mathscr{F}_j^{(e)} = \omega_{A/S,j}$$

by $\mathcal{O}_F$-stable $\mathcal{O}_S$-subbundles. We further require that each subquotient of the above filtrations is a locally free $\mathcal{O}_S$-module of rank one, and that it is annihilated by the action of $[\varpi_{\mathfrak{p}}]$ (cf. Subsection 2.2). We point out that in general the splitting model depends upon the choice of ordering of the p-adic embeddings of F. This dependence disappears when considering characteristic p fibers, so we ignore the issue in this introduction, but cf. Remark 2.3.

We observe that when A satisfies the Rapoport condition, so that the sheaf of invariant differentials $\omega_{A/S}$ is an invertible $(\mathcal{O}_S \otimes \mathcal{O}_F)$-module, there is exactly one possible filtration on each of the sheaves $\omega_{A/S,j}$, namely the one obtained by considering increasing powers of the uniformizer. Moreover, when $e = 1$, we have $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}} = \mathcal{M}_{\mathbb{F}}^{\mathrm{DP}} = \mathcal{M}_{\mathbb{F}}^{\mathrm{Ra}}$.

The subquotients $\omega_{j,l} := \mathscr{F}_j^{(l),\mathrm{univ}}/\mathscr{F}_j^{(l-1),\mathrm{univ}}$ of the universal filtration over $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$ define automorphic line bundles over the splitting model (cf. Notation 2.6), so that we have a good notion of Hilbert modular forms as (roughly) elements of

$$H^0\big(\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}, \bigotimes_{j,l} \omega_{j,l}^{\otimes k_{j,l}}\big).$$

To define suitable generalizations of the partial Hasse invariants, it is natural to look at the action of the Verschiebung map on the invariant differentials of the universal abelian scheme over $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$, and it is not difficult to see that the Verschiebung map preserves the filtrations $\mathscr{F}_j^{(l)}$ and hence induces homomorphisms:

$$V_j^{(l)} : \mathscr{F}_j^{(l)}/\mathscr{F}_j^{(l-1)} \to \big(\mathscr{F}_{j-1}^{(l)}/\mathscr{F}_{j-1}^{(l-1)}\big)^{(p)}.$$

Unfortunately, one can check that when $e > 1$ the zero locus of $V_j^{(l)}$ on $\mathcal{M}_{\mathbb{F}}^{\mathrm{PR}}$ is not irreducible. For example, when $e = 2$ and $f = 1$, the splitting model is obtained by blowing up the Deligne-Pappas moduli space in correspondence of its singularities (which are isolated points). The zero locus of the Verschiebung maps coincide with the union of the zero set of the (unique) classical partial Hasse invariant, together with the exceptional $\mathbb{P}^1$'s attached via the blow-ups.

In order to find a good notion of partial Hasse invariant on the splitting model, one needs to separate these irreducible components. To do so, in Section 3.1 we construct two types of "*generalized partial Hasse invariants*":

$$\text{when } l > 1, \quad m_j^{(l)} : \mathscr{F}_j^{(l)}/\mathscr{F}_j^{(l-1)} \to \mathscr{F}_j^{(l-1)}/\mathscr{F}_j^{(l-2)}, \text{ and}$$
$$\text{when } l = 1, \quad \mathrm{Hasse}_j : \mathscr{F}_j^{(1)} \to \big(\omega_{A/S,j-1}/\mathscr{F}_{j-1}^{(e-1)}\big)^{(p)},$$

where the first morphism is essentially given by multiplication by $[\varpi_{\mathfrak{p}}]$ (cf. Construction 3.3), and the second morphism is given by first "dividing by $[\varpi_{\mathfrak{p}}^{e-1}]$," and then applying the Verschiebung map (cf. Construction 3.6). As Hilbert modular forms, the partial Hasse invariant $m_j^{(l)}$ has weight $\omega_{j,l}^{\otimes -1} \otimes \omega_{j,l-1}$, while the partial Hasse invariant Hasse_j has weight $\omega_{j,1}^{\otimes -1} \otimes \omega_{j-1,e}^{\otimes p}$.

One can then see (cf. Lemma 3.8) that the map $V_j^{(l)}$ factors as the composition:

$$(m_{j-1}^{(l+1)})^{(p)} \circ \cdots \circ (m_{j-1}^{(e)})^{(p)} \circ \mathrm{Hasse}_j \circ m_j^{(2)} \circ \cdots \circ m_j^{(l)},$$

which explains the existence of the many irreducible components of the zero locus of $V_j^{(l)}$.