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OPTIMAL SOBOLEV REGULARITY OF ROOTS OF POLYNOMIALS

BY ADAM PARUSIŃSKI AND ARMIN RAINER

ABSTRACT. – We study the regularity of the roots of complex univariate polynomials whose coefficients depend smoothly on parameters. We show that any continuous choice of a root of a $C^{n-1,1}$-curve of monic polynomials of degree n is locally absolutely continuous with locally p-integrable derivatives for every $1 \leq p < n/(n-1)$, uniformly with respect to the coefficients. This result is optimal: in general, the derivatives of the roots of a smooth curve of monic polynomials of degree n are not locally $n/(n-1)$-integrable, and the roots may have locally unbounded variation if the coefficients are only of class $C^{n-1,\alpha}$ for $\alpha < 1$. We also prove a generalization of Ghisi and Gobbino's higher order Glaeser inequalities. We give three applications of the main results: local solvability of a system of pseudo-differential equations, a lifting theorem for mappings into orbit spaces of finite group representations, and a sufficient condition for multi-valued functions to be of Sobolev class $W^{1,p}$ in the sense of Almgren.

RÉSUMÉ. – Nous étudions la régularité des racines d'un polynôme complexe univarié dont les coefficients varient de façon lisse. Nous montrons que tout choix continu de racines d'une $C^{n-1,1}$-courbe de polynômes unitaires de degré n est localement absolument continu avec ses dérivées localement p-intégrables pour tout $1 \leq p < n/(n-1)$, uniformément par rapport aux coefficients. Ce résultat est optimal : en général, les dérivées de racines d'une courbe lisse de polynômes unitaires de degré n ne sont pas localement $n/(n-1)$-intégrables et la variation des racines peut être localement non bornée si les coefficients sont de classe $C^{n-1,\alpha}$ pour $\alpha < 1$. Nous montrons aussi une généralisation des inégalités de Glaeser d'ordre supérieur à la Ghisi et Gobbino. Nous donnons trois applications des résultats principaux : résolution locale d'un système d'équations pseudo-différentielles, un théorème de relèvement pour les applications à valeurs dans l'espace des orbites d'une représentation d'un groupe fini et une condition suffisante pour qu'une fonction multivaluée soit de classe de Sobolev $W^{1,p}$ au sens d'Almgren.

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1. Introduction

This paper is dedicated to the problem of determining the optimal regularity of the roots of univariate polynomials whose coefficients depend smoothly on parameters. There is a vast literature on this problem, but most contributions treat special cases:

- the polynomial is assumed to have only real roots ([9], [28], [45], [2], [21], [6], [7], [44], [8], [13], [30]),
- only radicals of functions are considered ([17], [11], [43], [12], [16]),
- it is assumed that the roots meet only of finite order, e.g., if the coefficients are real analytic or in some other quasianalytic class, ([10], [34], [35], [36], [39]),
- quadratic and cubic polynomials ([40]), etc.

In this paper we consider the general case: let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let

$$P_a(t)(Z) = P_{a(t)}(Z) = Z^n + \sum_{j=1}^{n} a_j(t) Z^{n-j}, \quad t \in (\alpha, \beta), \tag{1.1}$$

be a monic polynomial whose coefficients are complex valued smooth functions $a_j : (\alpha, \beta) \to \mathbb{C}$, $j = 1, \dots, n$. It is not hard to see that P_a always admits a continuous system of roots (e.g., [20, Ch. II Theorem 5.2]), but in general the roots cannot satisfy a local Lipschitz condition. For a long time it was unclear whether the roots of P_a admit locally absolutely continuous parameterizations. This question was affirmatively solved in our recent paper [32]: there is a positive integer $k = k(n)$ and a rational number $p = p(n) > 1$ such that, if the coefficients are of class C^k, each continuous root λ is locally absolutely continuous with derivative λ' being locally q-integrable for each $1 \le q < p$, uniformly with respect to the coefficients.

The problem of absolute continuity of the roots arose in the analysis of certain systems of pseudo-differential equations due to Spagnolo [41]; see Section 10.1. For the history of the problem we refer to the introduction of [32]. The main tool of [32] was the resolution of singularities. With this technique we could not determine the optimal parameters k and p.

1.1. Main results

In the present paper we prove the optimal result by elementary methods. Our main result is the following theorem.

Theorem 1. – *Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let P_a be a monic polynomial* (1.1) *with coefficients $a_j \in C^{n-1,1}([\alpha, \beta])$, $j = 1, \dots, n$. Let $\lambda \in C^0((\alpha, \beta))$ be a continuous root of P_a on (α, β). Then λ is absolutely continuous on (α, β) and belongs to the Sobolev space $W^{1,p}((\alpha, \beta))$ for every $1 \le p < n/(n-1)$. The derivative λ' satisfies*

$$\|\lambda'\|_{L^p((\alpha,\beta))} \le C(n, p) \max\{1, (\beta - \alpha)^{1/p}\} \max_{1 \le j \le n} \|a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}, \tag{1.2}$$

where the constant $C(n, p)$ depends only on n and p.

A well-known estimate for the Cauchy bound of a polynomial (cf. [27, p.56] or [33, (8.1.11)]) gives $|\lambda(t)| \le 2 \max_{1 \le j \le n} |a_j(t)|^{1/j}$ for all $t \in (\alpha, \beta)$, and hence

$$\|\lambda\|_{L^p((\alpha,\beta))} \le C(n)(\beta - \alpha)^{1/p} \max_{1 \le j \le n} \|a_j\|^{1/j}_{L^\infty((\alpha,\beta))}.$$

It follows that

$$\|\lambda\|_{W^{1,p}((\alpha,\beta))} \le C(n,p)\max\{1,(\beta-\alpha)^{1/p}\} \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}. \tag{1.3}$$

An application of Hölder's inequality yields the following corollary.

COROLLARY 1. – *Every continuous root of P_a on (α,β) is Hölder continuous of exponent $\gamma = 1-1/p < 1/n$, and*

$$\|\lambda\|_{C^{0,\gamma}([\alpha,\beta])} \le C(n,p)\max\{1,(\beta-\alpha)^{1/p}\} \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}. \tag{1.4}$$

Proof. – Indeed, $|\lambda(t)-\lambda(s)| \le |\int_s^t \lambda'\, d\tau| \le \|\lambda'\|_{L^p((\alpha,\beta))}|t-s|^{1-1/p}$. □

The result in Theorem 1 is best possible in the following sense:

- In general the roots of a polynomial of degree n cannot lie locally in $W^{1,n/(n-1)}$, even when the coefficients are real analytic. For instance, $Z^n = t$, $t \in \mathbb{R}$.
- If the coefficients are just in $C^{n-1,\delta}([\alpha,\beta])$ for every $\delta < 1$, then the roots need not have bounded variation in (α,β). See [16, Example 4.4].

A *curve* of complex monic polynomials (1.1) admits a continuous choice of its roots. This is no longer true if the dimension of the parameter space is at least two. In that case monodromy may prevent the existence of continuous roots. However, we obtain the following multiparameter result, where we impose the existence of a continuous root; see also Remark 8.

THEOREM 2. – *Let $U \subseteq \mathbb{R}^m$ be open and let*

$$P_a(x)(Z) = P_{a(x)}(Z) = Z^n + \sum_{j=1}^n a_j(x) Z^{n-j}, \quad x \in U, \tag{1.5}$$

be a monic polynomial with coefficients $a_j \in C^{n-1,1}(U)$, $j = 1,\dots,n$. Let $\lambda \in C^0(V)$ be a root of P_a on a relatively compact open subset $V \Subset U$. Then λ belongs to the Sobolev space $W^{1,p}(V)$ for every $1 \le p < n/(n-1)$. The distributional gradient $\nabla\lambda$ satisfies

$$\|\nabla\lambda\|_{L^p(V)} \le C(m,n,p,\mathscr{K}) \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}(\overline{W})}, \tag{1.6}$$

where $\mathscr{K}$ is any finite cover of $\overline{V}$ by open boxes $\prod_{i=1}^m(\alpha_i,\beta_i)$ contained in U and $W = \bigcup \mathscr{K}$; the constant $C(m,n,p,\mathscr{K})$ depends only on m, n, p, and the cover $\mathscr{K}$.

REMARK 1. – For any two distinct points x and y in V such that the segment $[x,y]$ is contained in V, the root λ satisfies a Hölder condition

$$\frac{|\lambda(x)-\lambda(y)|}{|x-y|^\gamma} \le C(m,n,p,\operatorname{diam}(V)) \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}([x,y])},$$

where $\gamma = 1-1/p < 1/n$. This follows easily from Theorem 2 and Remark 8.

The proof of Theorem 1 makes essential use of the recent result of Ghisi and Gobbino [16] who found the optimal regularity of radicals of functions (we will need a version for complex valued functions; see Section 3). But we independently prove and generalize Ghisi and Gobbino's higher order Glaeser inequalities (see Section 4.5) on which their result is based.