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A QUASI-LINEAR BIRKHOFF NORMAL FORMS METHOD. APPLICATION TO THE QUASI-LINEAR KLEIN-GORDON EQUATION ON $\mathbb{S}^1$

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Abstract. — Consider a nonlinear Klein-Gordon equation on the unit circle, with smooth data of size $\epsilon \to 0$. A solution u which, for any $\kappa \in \mathbb{N}$, may be extended as a smooth solution on a time-interval $]-c_\kappa \epsilon^{-\kappa}, c_\kappa \epsilon^{-\kappa}[$ for some $c_\kappa > 0$ and for $0 < \epsilon < \epsilon_\kappa$, is called an almost global solution. It is known that when the nonlinearity is a polynomial depending only on u, and vanishing at order at least 2 at the origin, any smooth small Cauchy data generate, as soon as the mass parameter in the equation stays outside a subset of zero measure of $\mathbb{R}_+^*$, an almost global solution, whose Sobolev norms of higher order stay uniformly bounded. The goal of this paper is to extend this result to general Hamiltonian *quasi-linear* nonlinearities. These are the only *Hamiltonian* non linearities that depend not only on u, but also on its space derivative. To prove the main theorem, we develop a Birkhoff normal form method for quasi-linear equations.

Résumé **(Une méthode de formes normales de Birkhoff quasi-linéaire. Application à l'équation quasi-linéaire de Klein-Gordon sur $\mathbb{S}^1$).** — Considérons une équation de Klein-Gordon non-linéaire sur le cercle unité, à données régulières de taille $\epsilon \to 0$. Appelons solution presque globale toute solution u, qui se prolonge pour tout $\kappa \in \mathbb{N}$ sur un intervalle de temps $]-c_\kappa \epsilon^{-\kappa}, c_\kappa \epsilon^{-\kappa}[$, pour un certain $c_\kappa > 0$ et $0 < \epsilon < \epsilon_\kappa$. Il est connu que de telles solutions existent, et restent uniformément bornées dans des espaces de Sobolev d'ordre élevé, lorsque la non-linéarité de l'équation est un polynôme en u nul à l'ordre 2 à l'origine, et lorsque le paramètre de masse de l'équation reste en dehors d'un sous-ensemble de mesure nulle de $\mathbb{R}_+^*$. Le but de cet article est d'étendre ce résultat à des non-linéarités *quasi-linéaires* Hamiltoniennes générales. Il s'agit en effet des seules non-linéarités *Hamiltoniennes* qui puissent dépendre non seulement de u, mais aussi de sa dérivée en espace. Nous devons, pour obtenir le théorème principal, développer une méthode de formes normales de Birkhoff pour des équations quasi-linéaires.

CONTENTS

CHAPTER 0

INTRODUCTION

The main objective of this paper is the construction of a Birkhoff normal forms method, applying to quasi-linear Hamiltonian equations. We use this method to obtain almost global solutions for quasi-linear Hamiltonian Klein-Gordon equations, with small Cauchy data, on the circle $\mathbb{S}^1$.

Let us first present the general framework we are interested in. Let Δ be the Laplace-Beltrami operator on $\mathbb{R}^d$ or on a compact manifold, and consider the evolution equation

$$(1)\qquad \begin{aligned}(\partial_t^2 - \Delta + m^2)v &= F(v, \partial_t v, \partial_x v, \partial_t\partial_x v, \partial_x^2 v)\\ v|_{t=0} &= \epsilon v_0\\ \partial_t v|_{t=0} &= \epsilon v_1,\end{aligned}$$

where v_0, v_1 are real valued smooth functions, $\epsilon > 0$ is small, F is a polynomial non-linearity with affine dependence in $(\partial_t\partial_x v, \partial_x^2 v)$, so that the equation is quasi-linear. We are interested in finding a solution defined on the largest possible time-interval when $\epsilon \to 0+$. If F vanishes at order $\alpha+1$ at the origin, local existence theory implies that the solution exists at least over an interval $]-c\epsilon^{-\alpha}, c\epsilon^{-\alpha}[$, if $v_0 \in H^{s+1}$, $v_1 \in H^s$ with s large enough, and that $\|v(t,\cdot)\|_{H^{s+1}} + \|\partial_t v(t,\cdot)\|_{H^s}$ stays bounded on such an interval. The problem we are interested in is the construction of almost global solutions, i.e. solutions defined on $]-c_\kappa\epsilon^{-\kappa}, c_\kappa\epsilon^{-\kappa}[$ for any κ.

This problem is well understood when one can make use of dispersion, e.g. when one studies (1) on $\mathbb{R}^d$, with v_0, v_1 smooth and quickly decaying at infinity (for instance $v_0, v_1 \in C_0^\infty(\mathbb{R}^d)$). When dimension d is larger or equal to three, Klainerman **[16]** and Shatah **[20]** proved independently global existence for small enough $\epsilon > 0$. Their methods were quite different: the main ingredient of Klainerman's proof was the use of vector fields commuting to the linear part of the equation. On the other hand, Shatah introduced in the subject normal form methods, which are classical tools in ordinary differential equations. Both approaches have been combined by Ozawa, Tsutaya and

Tsutsumi [**19**] to prove global existence for the same equation in two space dimensions. We also refer to [**10**] and references therein for the case of dimension 1.

A second line of investigation concerns equation (1) on a compact manifold (with a nonlinearity that may then depend also on x, even if we ignore this possible dependence in this introduction, for the sake of simplicity). In this case, no dispersion is available. Nevertheless, two trails may be used to obtain solutions, defined on time-intervals larger than the one given by local existence theory, and whose higher order Sobolev norms are uniformly bounded. The first one is to construct periodic or quasi-periodic (hence global) solutions. A lot of work has been devoted to these questions in dimension one, i.e. for $x \in \mathbb{S}^1$, when the non-linearity in (1) depends only on v. We refer to the work of Kuksin [**17, 18**], Craig and Wayne [**8**], Wayne [**21**], and for a state of the art around 2000, to the book of Craig [**7**] and references therein. More recent results may be found in the book of Bourgain [**6**]. Of course, this approach does not provide solutions to the Cauchy problem, as the traces at $t = 0$ of such quasi-periodic solutions do not exhaust the whole Sobolev space.

The second approach concerns the construction of almost global H^s-small solutions for the Cauchy problem (1) on $\mathbb{S}^1$, when the non-linearity depends only on v. In this case, small H^1 Cauchy data give rise to global solutions, and the question is to keep uniform control of the H^s-norm of the solution, over time-intervals of length $\epsilon^{-\kappa}$, for any κ and large enough s. This has been initiated by Bourgain [**5**], who stated a result of almost global existence and uniform control for $(\partial_t^2 - \partial_x^2 + m^2)v = F(v)$ on $\mathbb{S}^1$, when m stays outside a subset of zero measure, and the Cauchy data are small and smooth enough. A complete proof has been given by Bambusi [**1**], Bambusi-Grébert [**3**] (see also Grébert [**15**]). It relies on the use of a Birkhoff normal form method, exploiting the fact that when the non-linearity depends only on v, the equation may be written as a Hamiltonian system.

Let us mention that some of the results we described so far admit extensions to higher dimensions. Actually, constructions of periodic or quasi-periodic solutions for equations of type $(i\partial_t - \Delta + M)v = F(v)$ (where M is a convenient Fourier multiplier) or $(\partial_t^2 - \Delta + m^2)v = F(v)$ have been performed by Eliasson-Kuksin [**14**] and Bourgain [**6**] on higher dimensional tori. Almost global solutions for the Cauchy problem on spheres and Zoll manifolds have been obtained by Bambusi, Delort, Grébert and Szeftel [**2**] for almost all values of m.

We are interested here in the Cauchy problem when the non-linearity is a function not only of v, but also of derivatives of v. Recall that a non-linear wave equation is called semi-linear (resp. quasi-linear) if the non-linearity depends on derivatives up to order one (resp. up to order two and is linear in second order derivatives) of the unknown. Some results have been proved by Delort and Szeftel [**12, 13**] for semi-linear non-linearities of the form $F(v, \partial_t v, \partial_x v)$ on $\mathbb{S}^d$ or on Zoll manifolds. For instance, it