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THE 3D NAVIER-STOKES EQUATIONS SEEN AS A PERTURBATION OF THE 2D NAVIER-STOKES EQUATIONS

PAR DRAGOŞ IFTIMIE (*)

ABSTRACT. — We consider the periodic 3D Navier-Stokes equations and we take the initial data of the form $u_0 = v_0 + w_0$, where v_0 does not depend on the third variable. We prove that, in order to obtain global existence and uniqueness, it suffices to assume that $\|w_0\|_X \exp(\|v_0\|^2_{L^2(\mathbb{T}^2)}/C\nu^2) \leq C\nu$, where X is a space with a regularity H^δ in the first two directions and $H^{\frac{1}{2}-\delta}$ in the third direction or, if $\delta = 0$, a space which is L^2 in the first two directions and $B^{\frac{1}{2}}_{2,1}$ in the third direction. We also consider the same equations on the torus with the thickness in the third direction equal to ε and we study the dependence on ε of the constant C above. We show that if v_0 is the projection of the initial data on the space of functions independent of the third variable, then the constant C can be chosen independent of ε.

RÉSUMÉ. — LES ÉQUATIONS DE NAVIER-STOKES 3D VUES COMME UNE PERTURBATION DES ÉQUATIONS DE NAVIER-STOKES 2D. — On considère les équations de Navier-Stokes périodiques 3D et on prend la donnée initiale de la forme $u_0 = v_0 + w_0$, où v_0 ne dépend pas de la troisième variable. On démontre que, afin d'obtenir l'existence et l'unicité globale, il suffit de supposer que $\|w_0\|_X \exp(\|v_0\|^2_{L^2(\mathbb{T}^2)}/C\nu^2) \leq C\nu$, où X est un espace avec une régularité H^δ dans les deux premières directions et $H^{\frac{1}{2}-\delta}$ dans la troisième direction ou, si $\delta = 0$, un espace qui est L^2 dans les deux premières directions et $B^{\frac{1}{2}}_{2,1}$ dans la troisième direction. On considère aussi le même système sur le tore avec une épaisseur ε dans la troisième direction et on étudie la dépendance de ε de la constante C ci-dessus. On trouve que, si v_0 est la projection de la donnée initiale sur l'espace des fonctions indépendantes de la troisième variable, alors la constante C peut être choisie indépendante de ε.

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Introduction

The periodic 3D Navier-Stokes equations are the following:

$$\text{(N-S)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p, \\ \operatorname{div} u(t, \cdot) = 0 \quad \text{for all } t \geq 0, \\ u_{|t=0} = u_0. \end{cases}$$

Here, $u(t,x)$ is a periodic time-dependent 3-dimensional vector-field. For the sake of simplicity, we assume that the force is vanishing. This is not a serious restriction, it is clear that the difficulty in solving these equations comes from the non linear term. Similar results may be proved in the same way with a force square-integrable in time with values in the right space. The choice of periodic boundary conditions comes from the need to use the Fourier transform; for this reason our methods do not trivially extend to other classical boundary conditions.

It is well-known that in 2D, there exists a global unique solution for square-integrable initial velocity. In larger dimensions, unless some symmetry is assumed, global existence and uniqueness of solutions is known to hold only for small and more regular initial velocities. The goal of this paper is to prove global existence and uniqueness results by considering the 3D Navier-Stokes system as a perturbation of the 2D system. To do that, we write the initial data as the sum of a 2-dimensional initial part and a remainder. The main theorem says that, in order to obtain global existence, it suffices to assume the remainder small, and small compared to the 2-dimensional part.

Some stability results are already proved by G. Ponce, R. Racke, T.C. Sideris and E.S. Titi in [9] but the norm of the remainder is not estimated and the 2-dimensional part of the initial data is assumed to be in $H^1 \cap L^1$ and not in L^2, the optimal assumption. This loss of regularity appears when they take the product of a 2-dimensional function with a 3-dimensional function. This difficulty is overwhelmed here by introducing anisotropic spaces, where the variables are "separated". The loss of regularity is then optimal. Another advantage of these spaces is that they are larger than the usual Sobolev spaces, hence we obtain in the same time more general theorems.

It is natural to ask if the 3D Navier-Stokes equations on thin domains are close to the 2D Navier-Stokes equations from the point of view of global existence and uniqueness of solutions. A second aim of this work is to do the asymptotic study of the Navier-Stokes equations on $\mathbb{T}_\varepsilon = [0, 2\pi a] \times [0, 2\pi b] \times [0, 2\pi\varepsilon]$ when $\varepsilon \to 0$, as was first considered by G. Raugel and G.R. Sell [11], [10] and, afterwards, by J.D. Avrin [1], R. Temam and M. Ziane [12], [13] and I. Moise, R. Temam and M. Ziane [8]. By asymptotic study, we mean proving global existence and uniqueness of solutions for initial data in optimal sets, whose diameters should go to infinity when the slenderness of the domain goes to 0. To do that, it is natural to work in spaces where the third variable is distinguished. It appears that the anisotropic spaces are again well adapted to this study.

In an earlier paper [7], we proved global existence and uniqueness of solutions for (N-S) in $\mathbb{R}^3$ with small initial data in

$$\mathcal{H}^{\delta_1,\delta_2,\delta_3}, \quad \delta_1+\delta_2+\delta_3=\frac{1}{2}, \quad -\frac{1}{2}<\delta_i<\frac{1}{2},$$

a space which is H^{δ_i} in the i-th direction. Here we apply in the periodic case the work we have done there. The precise result is that there exists a positive constant C, independent of ν, such that if $0<\delta<1$ and the initial data is v_0+w_0 with v_0 independent of the third variable, then, in order to obtain global existence and uniqueness of solutions, it suffices to assume that

$$\|w_0\|_X \exp\Big(\frac{\|v_0\|^2_{L^2(\mathbb{T}^2)}}{C\nu^2}\Big) \le C\nu, \tag{0.1}$$

where X is a space which is H^δ in the first two variables and $H^{\frac{1}{2}-\delta}$ in the third variable, or, if $\delta=0$, a space which is L^2 in the first two variables and $B^{\frac{1}{2}}_{2,1}$ in the third variable, where $B^s_{p,q}$ is the usual Besov space given by

$$B^s_{p,q}=\big\{u\in\mathcal{S}' \text{ such that } \big\|\,2^{is}\|\Delta_i u\|_{L^p}\big\|_{\ell^q}<\infty\big\},$$

where $\Delta_i u$ is defined in (1.1). We shall also prove local existence and uniqueness of solutions for arbitrary initial data in the spaces above.

In the third paragraph we work in $\mathbb{T}_\varepsilon$ and we study the dependence on ε of the constant of inequality (0.1). We shall prove that if v_0 is the projection of the initial data on the space of functions independent of x_3 and $0<\delta\le\frac{1}{2}$, then the constant C can be chosen independent of ε. This will imply that global existence and uniqueness is achieved as long as

$$\|w_0\|_{H^{\frac{1}{2}}(\mathbb{T}_\varepsilon)} \exp\Big(\frac{\|v_0\|^2_{L^2(\mathbb{T}^2)}}{C\nu^2}\Big) \le C\nu. \tag{0.2}$$

The inequality above can be read in various ways. For instance, it is implied by

$$\|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \exp\Big(\frac{\|v_0\|^2_{L^2(\mathbb{T}^2)}}{C\nu^2}\Big) \le C\nu\varepsilon^{-\frac{1}{2}},$$

or, for all $\alpha\ge 0$, by

$$\|v_0\|_{L^2(\mathbb{T}^2)} \le C\nu\big(1+\sqrt{-\alpha\log\varepsilon}\,\big) \quad\text{and}\quad \|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \le C\nu\varepsilon^{-\frac{1}{2}+\alpha}.$$

Finally, if one needs to have a larger v_0, one can take v_0 arbitrarily large, the price to pay is that w_0 has to be assumed exponentially small with respect to that v_0.

Let us compare this theorem with the previous results.

The precise results of G. Raugel and G.R. Sell [11], [10] are rather complicated so we give only an approximation: they consider various boundary conditions and obtain global existence and uniqueness of solutions as long as

$$\|v_0\|_{H^1(\mathbb{T}^2)} \le C\varepsilon^{-5/24} \quad \text{and} \quad \|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \le C\varepsilon^{-5/48}$$

or

$$\|v_0\|_{H^1(\mathbb{T}^2)} \le C\varepsilon^{-17/32}, \ \|v_0^3\|_{L^2(\mathbb{T}^2)} \le C\varepsilon^{\frac{1}{2}} \ \text{and} \ \|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \le C\varepsilon^{-1/8},$$

where v_0^3 is the third component of v_0.

In the paper of J.D. Avrin [1] it is shown that $\|u_0\|_{H^1} \le C\lambda_1^{-1/4}$ suffices in the case of homogeneous Dirichlet boundary conditions; we denoted by λ_1 the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions. In the case of a thin domain, the equivalent of Avrin's result would be:

$$\|u_0\|_{H^1} \le C\varepsilon^{-\frac{1}{2}}.$$

Let us note that in the case of homogeneous Dirichlet boundary conditions the 2-dimensional part can not be defined, so one of the major difficulties of the problem, mixture of 2D functions with 3D functions, does not appear.

I. Moise, R. Temam and M. Ziane [8] prove that it is sufficient to assume that

$$\|v_0\|_{H^1(\mathbb{T}^2)} \le C\varepsilon^{-\frac{1}{3}+\delta} \quad \text{and} \quad \|w_0\|_{H^1(\mathbb{T}_\varepsilon)} \le C\varepsilon^{-\frac{1}{6}+\delta},$$

where δ is a positive number.

Finally we mention that spherical domains are considered by R. Temam and M. Ziane [13].

1. Notations and preliminary results

Many of the notations and the results from [7] remain valid here with minor modifications; for those results, we shall only sketch the proofs. The main differences are that we use the Littlewood-Paley theory in two variables instead of three and we have to adjust to the periodic case the definition of the Δ_q operators. We work in

$$\mathbb{T}^3 = [0,2\pi] \times [0,2\pi] \times [0,2\pi]$$

and we denote by $(x_1,x_2,x_3) = (x',x_3)$ the variable in $\mathbb{T}^3$. All the functions are assumed to have vanishing integral on $\mathbb{T}^3$. Let

$$L^{p,q} = \big\{u \text{ such that } \|u\|_{L^{p,q}} \stackrel{\text{def}}{=} \big\|\|u(x)\|_{L^q_{x_3}}\big\|_{L^p_{x'}} < \infty\big\},$$

and $\ell^{p,q}$ be the similar space for sequences. Obviously, when $p=q$, the spaces $\ell^{p,p}$ and $L^{p,p}$ are nothing else but the usual ℓ^p and L^p spaces. The order of integrations is important, as shown by the following remark (see [7]):