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THETA HEIGHT AND FALTINGS HEIGHT

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ABSTRACT. — Using original ideas from J.-B. Bost and S. David, we provide an explicit comparison between the Theta height and the stable Faltings height of a principally polarized Abelian variety. We also give as an application an explicit upper bound on the number of K-rational points of a curve of genus $g \geq 2$ under a conjecture of S. Lang and J. Silverman. We complete the study with a comparison between differential lattice structures.

RÉSUMÉ (*Hauteur Thêta et hauteur de Faltings.*)— On propose dans cet article les détails d'une preuve de comparaison explicite entre la hauteur Thêta et la hauteur de Faltings stable d'une variété abélienne principalement polarisée et définie sur un corps de nombres K. Cette preuve est basée sur les idées de J.-B. Bost et S. David. On trouvera de plus le calcul d'une borne explicite sur le nombre de points K-rationnels d'une courbe de genre $g \geq 2$ en supposant une conjecture de S. Lang et J. Silverman. Ce travail est complété par une comparaison entre plusieurs structures de réseaux sur l'espace tangent en 0.

1. Introduction

Let (A, L) be a principally polarized Abelian variety defined over a number field K. The aim of the article is to compare the Theta height $h_\Theta(A, L)$ of Definition 2.6, and the (stable) Faltings height $h_F(A)$ of Definition 2.1. These

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two ways of defining the height of an Abelian variety are both of interest, and the fact that they can be precisely compared can be very helpful. For instance, several conjectures are formulated with the Faltings height because it does not depend on the projective embedding of A that you may choose, but one may fix an ample and symmetric line bundle on A and study the Theta height associated when one seeks more effectivity (see for example [9] or [28], and also [27]); let us stress that these ways of defining the height of an Abelian variety are very natural: the Theta height is a height on the moduli space of principally polarized Abelian variety and the Faltings height is a height on the moduli space (stack) of Abelian varieties (without polarization), but with a metric with logarithmic singularities (see the definitions below and refer to [22] for the Theta height, [19] and [13] for the Faltings height).

The ideas needed to explicitly compute the constants of comparison between these heights were given by Bost and David in a letter to Masser and Wüstholz [5]. Here is the strategy: using the theory of Moret-Bailly-models we express the Néron-Tate height of a point $P \in A(K)$ in terms of the Theta height of P, the Faltings height of A and some base point contributions (see Lemma 5.2). Then we take $P = O$ and we estimate the base point contributions via vector bundles inclusions and theta functions analysis. We give here the arguments, the constants and several complements, concerning the Lang-Silverman conjecture for instance. We also complete this work by giving in Section 6 an explicit comparison between several differential lattice structures associated to A, see the end of this introduction.

One should underline that this explicit comparison gives also a direct proof of the fact that the Faltings height is actually a height (*i.e.* verifies the Northcott property), see the Remark 1.4 below for a lower bound. Arguments for proving that h_F is a height can be found in the original article [12] and in [13]. See also the Theorem 1.1 page 115 of [19] (seminar [32]); the idea is to compactify some moduli schemes and to compare the stable Faltings height of an Abelian variety to the projective height (with logarithmic singularities) of the corresponding point in the moduli space. There is another proof given by Moret-Bailly in Theorem 3.2 page 233 of [20] using the "formule clef" 1.2 page 190. See also the Theorem 2.1 given in [4] page 795-04, where the proof relies on some estimates of the "rayon d'injectivité".

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We use the notations $\mathfrak{S}_g$ for the Siegel space and $\mathfrak{F}_g$ for the fondamental domain, both defined in §2.1. We add a Theta structure of level r (see §2.3), where $r > 0$ is an even integer. With these notations, we get the following theorem.

THEOREM 1.1. — *Let A be an Abelian variety of dimension g, defined over $\overline{\mathbb{Q}}$, equipped with a principal polarization defined by a symmetric ample line bundle L on A. Let K be a number field such that A and L may be defined over K. For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, let $\tau_\sigma \in \mathfrak{F}_g$ such that there exists an isomorphism between principally polarized complex Abelian varieties $A_\sigma(\mathbb{C}) \simeq \mathbb{C}^g/(\mathbb{Z}^g + \tau_\sigma \mathbb{Z}^g)$. Then, the following inequalities hold:*

$$m(r,g) \leq h_\Theta(A,L) - \frac{1}{2}h_F(A) - \frac{1}{4[K:\mathbb{Q}]}\sum_{\sigma:K\hookrightarrow\mathbb{C}} \log(\det(\operatorname{Im}\tau_\sigma)) \leq M(r,g)\,.$$

Above, $m(r,g)$ and $M(r,g)$ denote constants depending only on the level r and the dimension g. More precisely, if we take:

$$m(r,g) = g\left[\frac{1}{4}\log(4\pi) - \frac{1}{2}r^{2g}\log(r)\right],$$

$$M(r,g) = \frac{g}{4}\log(4\pi) + g\log(r) + \frac{g}{2}\log\left(2 + \frac{2}{3^{\frac{1}{4}}}2^{\frac{g^3}{4}}\right),$$

then the result holds.

REMARK 1.2. — According to the so called Matrix Lemma of Masser (see [17] page 115 or [18] page 436) there exists a constant $C(g)$ such that under the hypothesis of the above theorem:

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\sigma:K\hookrightarrow\mathbb{C}} \left|\log\Big(\det(\operatorname{Im}\tau_\sigma)\Big)\right| \leq C(g)\log\Big(\max\{h_\Theta(A,L),1\}+2\Big)\,.$$

Using the article [9] page 697 and a few calculations it is possible to prove such a bound with the explicit constant $C(g) = \frac{8g}{\pi}(1+2g^2\log(4g))$. See also [14] Lemma 2.12 page 99 for a similar statement involving the Faltings height.

Thus, we shall establish in § 5.2.1 the following versions of Faltings' estimate (see [12]):

COROLLARY 1.3. — *For every integer $g \geq 1$ and even integer $r \geq 2$, there exists effectively computable constants $C_1(g,r)$, $C_2(g,r)$, $C_3(g,r)$ depending only on g and r such that the following holds. Let A be an Abelian variety of dimension g defined over $\overline{\mathbb{Q}}$, equipped with a principal polarisation defined by some symmetric ample line bundle L on A. Let $h_\Theta = \max\{h_\Theta(A,L),1\}$ and $h_F = \max\{h_F(A),1\}$. Then, one has:*

1. $\left|h_\Theta(A,L) - \frac{1}{2}h_F(A)\right| \leq C_1(g,r)\log\Big(h_\Theta + 2\Big),$

2. $\left|h_\Theta - \frac{1}{2}h_F\right| \leq C_2(g,r)\log\Big(\min\Big\{h_\Theta, h_F\Big\}+2\Big)$,

3. $\left|h_\Theta(A,L) - \frac{1}{2}h'_F(A)\right| \leq C_3(g,r)$,

where $h'_F(A)$ is a modified Faltings height of A, defined in 2.2. More precisely, the above relations hold with:

$$C_1(g,r) = C_3(g,r) = 6r^{2g}\log(r^{2g}) \text{ and } C_2(g,r) = 1000r^{2g}(\log(r^{2g}))^5.$$

Remark 1.4. — For an Abelian variety A of dimension g and level structure r, the inequality of Theorem 1.1 and the Remark 1.2 give after a short calculation:

$$h_F(A) \geq -C(g)\log C(g) - M(r,g)\ ,$$

where $M(r,g) = \frac{g}{4}\log(4\pi) + g\log(r) + \frac{g}{2}\log\Big(2 + \frac{2}{3^{\frac{1}{4}}}2^{\frac{g^3}{4}}\Big)$ and $C(g) = \frac{8g}{\pi}\left(1+2g^2\log(4g)\right)$. One could expect a better constant, see Bost in [2] page 6 who gives: $h_F(A) \geq -g\log(2\pi)/2$.

Remark 1.5. — The inequalities (1) and (3) both hold if one replaces $h_\Theta(A,L)$, $h_F(A)$ and $h'_F(A)$ respectively by $h_\Theta = \max\{h_\Theta(A,L),1\}$, $h_F = \max\{h_F(A),1\}$ and $h'_F = \max\{h'_F(A),1\}$ in the left hand sides.

Remark 1.6. — One can notice that the bounds are sharper for small r, so in practice one will often take $r=2$ or $r=4$.

We now give the example of a difficult conjecture by Lang and Silverman stated with the Faltings height. It was originally a question by Lang concerning elliptic curves, and was generalised by Silverman afterwards. As a matter of fact, if we combine the inequality of this conjecture with the work of David and Philippon [9] and the work of Rémond [28], we get a new explicit bound on the number of rational points on curves of genus $g \geq 2$, provided that we can explicitely compare the Faltings height that appears in the conjecture and the Theta height that appears in the calculations of [9] and [28]. To be concise, one can say that an explicit Lang-Silverman inequality would give an explicit upper bound on the number of rational points on a curve of genus $g \geq 2$ independant of the height of the Jacobian of the curve (but still depending on the Mordell-Weil rank of the Jacobian).

First recall the original conjecture of Silverman ([30] page 396):

Conjecture 1.7 (Lang-Silverman version 1). — *Let $g \geq 1$ be an integer. For any number field K, there exists a positive constante $c(K,g)$ such that for any Abelian variety A/K of dimension g, for any ample and symmetric line bundle L on A and for any point $P \in A(K)$ such that $\mathbb{Z}\cdot P$ is Zariski-dense, one has:*

$$\widehat{h}_{A,L}(P) \geq c(K,g)\ \max\Big\{h_F(A/K),\ 1\Big\}\ ,$$