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MILNOR NUMBER OF PAIRS AND PENCILS OF PLANE HOLOMORPHIC GERMS

by Adrian Szawlowski

Abstract. — We introduce a Milnor number of pairs of plane holomorphic germs and investigate the relation between the Milnor numbers in a pencil of such functions.

Résumé (*Le nombre de Milnor des paires et des pinceaux de germes holomorphes plans*)

Nous introduisons un nombre de Milnor pour les paires de germes holomorphes plans et nous étudions la relation entre les nombres de Milnor apparaissant dans un tel pinceau.

1. Introduction

The study of germs of holomorphic functions $f\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ has very much progressed since the work of Milnor ([13]). Among other things he proved a result which is nowadays called Milnor's fibration theorem and has relations with knot theory. There is another fibration, which is also referred very often to as Milnor fibration (or Milnor-Lê fibration) and is equivalent to the previous one. Some decades later, the interest in meromorphic germs has raised. For example both fibration theorems have extensions to the meromorphic case (compare [1] for the first fibration and [16] resp. [8] for the second fibration

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and [2] for their comparison in the semitame case). It seems however that the investigation of meromorphic germs is still not complete at all. Another way to perceive them is for example by considering a pencil spanned by the two given germs. What we do here is to look at the Milnor numbers that occur in such a pencil. The basic invariant of a holomorphic germ $f\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ with an isolated critical point at the origin is its Milnor number

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle \partial_x f, \partial_y f\rangle}.$$

Given another holomorphic germ $g\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ we form the pencil generated by f and g which associates to each $(s:t) \in \mathbb{P}^1$ the function germ $sf+tg$. In this paper we address the question whether there is a relation between all the Milnor numbers of the pencil members. Apart from some minor propositions, the main results in this paper are Theorem 2.6, Theorem 3.3, Conjecture 4.1 and its special case Proposition 4.4.

As is standard we denote by $\mathbb{C}\{x,y\}$ the ring of germs of holomorphic functions at the origin, or equivalently the complex power series ring at the origin and by $\mathfrak{m}$ its maximal ideal. For two germs $f,g \in \mathbb{C}\{x,y\}$ we denote by $i(f,g)$ their intersection number at the origin of $\mathbb{C}^2$. For an ideal I in $\mathbb{C}\{x,y\}$, we denote by $V(I)$ its vanishing locus, the germ of an analytic set at the origin in $\mathbb{C}^2$. Finally we denote by ∂ the partial derivative after x and/or y.

I like to thank the referee for useful comments, especially regarding the last paragraph.

2. Milnor Number of Pairs

For any $f,g \in \mathbb{C}\{x,y\}$ we introduce the notation $\omega(f,g)$ for the ideal in $\mathbb{C}\{x,y\}$ generated by the elements $f_x g - f g_x$ and $f_y g - f g_y$. It is clear that $\omega(f,g)$ is a subideal of $\langle f,g\rangle$. Sometimes if f and g are clear from the context, we simply write ω instead of $\omega(f,g)$. Let us now define the following possibly infinite numbers

$$\mu(f,g) := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\omega(f,g)}, \tag{2.1}$$

$$\nu(f,g) := \dim_{\mathbb{C}} \frac{\langle f,g\rangle}{\omega(f,g)}. \tag{2.2}$$

Clearly we have $\mu(f,g) = \nu(f,g) + i(f,g)$. At an arbitrary point $p \in \mathbb{C}^2$ we define $\mu_p(f,g)$ by using the local power series ring $\mathcal{O}_p$ and the ideal $\omega_p(f,g)$ at p in the obvious way:

$$\mu_p(f,g) := \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{\omega_p(f,g)}. \tag{2.3}$$

As will follow from the propositions below, $\mu(f,g)$ is the analogue to the classical Milnor number $\mu(f)$ of a holomorphic germ and will be referred to as Milnor number of pairs or meromorphic Milnor number in the following.

The next propositions give first information on these numbers.

PROPOSITION 2.1. — *Let $f,g \in \mathbb{C}\{x,y\}$. Then for any coordinate transformation $\Phi \in \mathrm{Aut}(\mathbb{C}^2,\mathbf{0})$ and any unit $u \in \mathbb{C}\{x,y\}$ we have $\mu(u\cdot f\circ\Phi, u\cdot g\circ\Phi) = \mu(f,g)$ and the analogous property holds for ν.*

This has an elementary proof using derivatives. Easy as well is the following

PROPOSITION 2.2. — *Let $f,g \in \mathbb{C}\{x,y\}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\,\mathbb{C}$. Then*

$$\mu(af+bg, cf+dg) = \mu(f,g)$$

and the same property holds for ν.

The next proposition shows that our generalization of the Milnor number includes as a special case the classical Milnor number.

PROPOSITION 2.3. — *Let $f,g \in \mathbb{C}\{x,y\}$. If $f(\mathbf{0}) \neq 0$ but $g(\mathbf{0}) = 0$, then $\mu(f,g) = \nu(f,g) = \mu(g)$ (maybe infinite).*

Proof. — It is clear that if $f(\mathbf{0}) \neq 0$, we have $\mu(f,g) = \nu(f,g)$. If we assume $f(\mathbf{0}) \neq 0$ then the ideals $\omega, f^{-2}\omega$ are the same and the latter can be written as $\langle \partial(g/f)\rangle$. Hence $\mu(f,g) = \mu(g/f)$. By the invariance of the Milnor number under the action of the contact group we obtain $\mu(f,g) = \mu(g)$. □

It is well-known that the classical Milnor number of a holomorphic germ $f \in \mathfrak{m}$ is equal to one if and only if f is critical and Morse at the origin. The following proposition shows that the situation with the new Milnor number is also very pleasant.

PROPOSITION 2.4. — *Let $f,g \in \mathbb{C}\{x,y\}$ with $f(\mathbf{0})g(\mathbf{0}) = 0$. Then $\mu(f,g) = 1$ if and only if we are in one of the following cases:*

- *We have $f(\mathbf{0}) \neq 0$ and g/f vanishes at the origin, is critical and Morse there.*
- *We have $g(\mathbf{0}) \neq 0$ and f/g vanishes at the origin, is critical and Morse there.*
- *The pair (f,g) is right equivalent to (x,y).*

Proof. — Let $\mu(f,g) = 1$. Since $\mu(f,g) = \nu(f,g) + i(f,g)$ two cases are possible:

- $\nu(f,g) = 0, i(f,g) = 1$ or
- $\nu(f,g) = 1, i(f,g) = 0$.