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GROUP SCHEMES WITH $\mathbb{F}_q$-ACTION

BY THOMAS POGUNTKE

ABSTRACT. — Via a construction due to V. Drinfel′d, we prove an equivalence of categories, generalizing the equivalence between commutative flat group schemes in characteristic p with trivial Verschiebung and their Dieudonné modules to group schemes with $\mathbb{F}_q$-action.

RÉSUMÉ (*Schémas en groupes avec $\mathbb{F}_q$-action*). — Au moyen d'une construction par V. Drinfel′d, nous prouvons une équivalence de catégories, généralisant l'équivalence entre les schémas en groupes plats commutatifs en caractéristique p annulé par le décalage et leurs $\mathbb{F}_p$-modules de Dieudonné aux schémas en groupes avec action de $\mathbb{F}_q$.

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1. Introduction

Let p be a prime and let k be a field of characteristic p. Denote by Gr_k^+ the category of affine commutative group schemes over k which can be embedded into $\mathbb{G}_a^N$ for some set N. We assign to $G \in \mathrm{Gr}_k^+$ its Dieudonné $\mathbb{F}_p$-module $\mathcal{M}(G) = \mathrm{Hom}_{\mathrm{Gr}_k^+}(G, \mathbb{G}_a)$, with the obvious left module structure over $\mathrm{End}_{\mathrm{Gr}_k^+}(\mathbb{G}_a) \cong k[F]$, the non-commutative polynomial ring with

$$F\lambda = \lambda^p F \text{ for } \lambda \in k.$$

These Dieudonné modules completely classify group schemes of the above type, as shown by the following theorem.

Theorem 1.1 ([2], IV, §3, 6.7). — *The contravariant functor $\mathcal{M}$ defines an exact anti-equivalence of categories*

$$\mathcal{M} : \mathrm{Gr}_k^+ \longrightarrow k[F]\text{-Mod}. \tag{1.1}$$

Under this duality, group schemes of finite presentation correspond to finitely generated $k[F]$-modules, and finite group schemes to finite-dimensional k-vector spaces.

The above result allows us to describe the structure of our category over a perfect field, and its simple objects if k is algebraically closed.

Theorem 1.2 ([2], IV, §3, 6.9). — *Let k be a perfect field. Then $G \in \mathrm{Gr}_k^+$ is algebraic if and only if it can be written as a product*

$$G \cong \mathbb{G}_a^n \times \pi_0(G) \times H,$$

where $n \in \mathbb{N}$, H is a finite product of group schemes of the form α_{p^s}, and $\pi_0(G)$ is an étale sheaf of finite $\mathbb{F}_p$-vector spaces. If k is algebraically closed, then

$$\pi_0(G) \cong (\underline{\mathbb{F}_p})^m, m \in \mathbb{N}.$$

On the other hand, let S be a scheme of characteristic p. Consider the category $\mathrm{gr}_S^{+\vee}$ of flat group schemes locally of finite presentation over S of height ≤ 1 (i.e., killed by their Frobenius). Let $p\text{-}\mathrm{Lie}_S$ denote the category of finite locally free $\mathcal{O}_S$-p-Lie algebras. Then we have the following classification theorem.

Theorem 1.3 ([7], Remark 7.5). — *The covariant functor*

$$\mathcal{L} : \mathrm{gr}_S^{+\vee} \longrightarrow p\text{-}\mathrm{Lie}_S, G \longmapsto \mathrm{Lie}(G),$$

defines an equivalence of categories.

Two of our main results generalize Theorem 1.1, and reduce (for "$q = p$") to Theorem 1.3 via Cartier duality, respectively. Moreover, we formulate two conjectures under which they unify.

Assume that S is an $\mathbb{F}_q$-scheme for some prime power $q = p^r$. Our group schemes G are affine, commutative, flat over S and carry an $\mathbb{F}_q$-action. We require that locally on S, there is an embedding $G \hookrightarrow \mathbb{G}_a^N$ for some set N, which respects the $\mathbb{F}_q$-actions.

The category of these group schemes will be denoted by $\mathbb{F}_q\text{-}\operatorname{Gr}_S^+$, and its full subcategory of finite group schemes of finite presentation is called $\mathbb{F}_q\text{-}\operatorname{gr}_S^+$.

On the other hand, we consider left $\mathcal{O}_S[F^r]$-modules, which are flat as $\mathcal{O}_S$-modules. They are called $\mathbb{F}_q$-shtukas over S, and their category is denoted by $\mathbb{F}_q\text{-}\operatorname{Sht}_S$. We write $\mathbb{F}_q\text{-}\operatorname{sht}_S$ for the full subcategory of $\mathbb{F}_q\text{-}\operatorname{Sht}_S$ of locally free modules of finite rank over $\mathcal{O}_S$.

We study the following generalization of the contravariant functor (1.1),

$$\mathcal{M}_q = \mathcal{M} : \mathbb{F}_q\text{-}\operatorname{gr}_S^+ \longrightarrow \mathbb{F}_q\text{-}\operatorname{sht}_S, G \longmapsto \operatorname{Hom}_{\mathbb{F}_q\text{-}\operatorname{Gr}_S^+}(G, \mathbb{G}_a).$$

We also explain the construction of a functor in the other direction,

$$\mathcal{G}_q = \mathcal{G} : \mathbb{F}_q\text{-}\operatorname{sht}_S \longrightarrow \mathbb{F}_q\text{-}\operatorname{gr}_S^+,$$

which is fully faithful and left-adjoint to $\mathcal{M}$. However, $\mathcal{G}_q$ does not define an equivalence of categories for $q \neq p$. Rather, we describe a full subcategory $\mathbb{F}_q\text{-}\operatorname{gr}_S^{+,\mathrm{b}}$ of *balanced* group schemes in $\mathbb{F}_q\text{-}\operatorname{gr}_S^+$, and prove that it is the essential image of $\mathcal{G}$.

Namely, let $G = \operatorname{Spec}(B_G) \in \mathbb{F}_q\text{-}\operatorname{gr}_S^+$. We show that the space of primitive elements in B_G decomposes into eigenspaces for the $\mathbb{F}_q^\times$-action as

$$\operatorname{Prim}(B_G) = \bigoplus_{s=0}^{r-1} \operatorname{Prim}_{p^s}(B_G). \tag{1.2}$$

We call G balanced, if the p-Frobenii $\operatorname{Prim}_{p^t}(B_G) \to \operatorname{Prim}_{p^{t+1}}(B_G), x \mapsto x^p$, are bijective for all $0 \leq t < r-1$. Note that when $q = p$, we recover $\mathbb{F}_p\text{-}\operatorname{gr}_S^{+,\mathrm{b}} = \operatorname{gr}_S^+$.

THEOREM 1.4. — *The functor $\mathcal{G} : \mathbb{F}_q\text{-}\operatorname{sht}_S \to \mathbb{F}_q\text{-}\operatorname{gr}_S^{+,\mathrm{b}}$ defines an exact anti-equivalence of categories with quasi-inverse $\mathcal{M}$.*

Our definition of the balanced subcategory of $\mathbb{F}_q\text{-}\operatorname{gr}_S^+$ is inspired by Raynaud's paper [15]. He considers finite commutative group schemes G with an action of $\mathbb{F}_q$, and the decomposition of the augmentation ideal into eigenspaces for the $\mathbb{F}_q^\times$-action,

$$I_G = \bigoplus_{j=1}^{q-1} I_j,$$

similarly to (1.2). Note that all summands I_j are finite locally free $\mathcal{O}_S$-modules. Raynaud imposes the condition that $\operatorname{rk}(I_j) = 1$, for all j.

We define a group scheme $G \in \mathbb{F}_q\text{-}\operatorname{gr}_S^+$ to be *quasi-balanced* if $\operatorname{rk}(I_j)$ is the same for all j. This turns out to be almost the same as being balanced; in