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THE JONES-WITTEN INVARIANTS OF KNOTS

par **Michael ATIYAH**

1. INTRODUCTION

One of the most remarkable developments of recent years has been the work initiated by Vaughan Jones [2] [3] on knot invariants. This has all the hallmarks of great mathematics. It produces simple new invariants which solve classical problems and it involves a very wide range of ideas and techniques from practically all branches of mathematics and physics. Here is a list of the areas which have been significantly involved in the theory up to the present : combinatorics, group representations, algebraic geometry, differential geometry, differential equations, topology, Von Neumann algebras, statistical mechanics, quantum field theory. Moreover the subject continues to develop rapidly and a final picture has not yet emerged.

Given this very wide field I have to be very selective for a one-hour presentation. I will concentrate on some aspects and I shall have to omit all the technicalities. Moreover, to shorten the exposition, I will discuss only the simplest case of the theory. Fuller accounts can be found in the papers of Vaughan Jones [3] and Witten [9].

In 1984 Vaughan Jones surprised the experts in knot theory by producing a *polynomial invariant*, now known as the Jones polynomial $V(q)$, which was superficially similar to the classical Alexander polynomial but was, in essential features, rather different. In particular $V(q)$ could distinguish (some) knots from their mirror images. For this and other reasons $V(q)$ turned out to be a very effective tool in knot theory and, as a result, old conjectures of P.G. Tait from the 19th century have now been established.

The Jones polynomial can be profitably studied from many angles and it has been generalized in several ways to produce further knot invariants. Much of this work has involved important ideas from theoretical physics, essentially physics of 2 dimensions. However a major break-through came in 1988 when Witten [10] gave a simple interpretation of the Jones polynomial in terms of 3-dimensional physics. These ideas of Witten are based on a heuristic use of the Feynman integral, but they lead to very explicit results and calculations which can be verified by alternative rigorous methods. A full mathematical treatment of Witten's theory has yet to appear, so my account has to be somewhat sketchy and incomplete.

Not only does Witten's theory provide a physical "meaning" for the Jones invariants but it also extends them to knots in an arbitrary compact oriented 3-manifold. This is a major generalization which had been attempted unsuccessfully via other methods. Finally, and most significantly, Witten's generalization allows us to define "relative invariants", for 3-manifolds with boundary. In this case the invariants are not numbers but take their values in a vector space associated with the boundary. This facility, of allowing manifolds with boundary, makes the theory much more flexible and greatly facilitates computation, even for the "absolute" case of closed 3-manifolds. The situation may roughly be compared with the story of Lefschetz numbers in classical topology. The number of fixed points of a self-map (analogous to the Jones invariant of a knot) re-interpreted as the Lefschetz number, through the induced map on homology, becomes part of a larger theory (analogous to Witten's theory) and hence more computable.

In the next section I will summarize the key features of the Jones polynomial, before going on in section 3 to describe Witten's theory. In section 4 I will outline the way in which Witten's theory may be developed mathematically. I will make no attempt in this presentation to give the physical interpretation via Feynman integrals. For this I refer to Witten's papers [9] [10]. For a general survey of "topological quantum field theories" see also [1] [8].

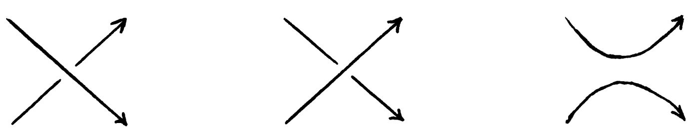
2. THE JONES POLYNOMIAL

We shall deal with oriented knots and links. These are just oriented 1-dimensional submanifolds of the 3-space S^3 : a knot being the case of one component. For an oriented link L the Jones polynomial $V_L(q)$ is a finite Laurent series in the variable $q^{\frac{1}{2}}$ with integer coefficients. Its first basic properties are :

(2.1) $V_L(q) = 1$ when L in the standard unknotted circle,

(2.2) $V_{L^\star}(q) = V_L(q^{-1})$ where $L^\star$ is the mirror image of L .

$V_L(q)$ can be characterized by a *skein relation.* For this we consider a generic plane projection of L , so that all crossing points have just 2 branches, one "over" and one "under". Focussing attention on one crossing point we can then consider the 3 versions of L obtained by allowing the 3 different possibilities as shown below :



L_+ $\qquad$ L_- $\qquad$ L_0

The skein relation for $V_L(q)$ is the linear relation :

$$q^{-1}V_{L_+}(q) - qV_{L_-}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\ V_0(q) \tag{2.3}$$

It is not hard to show that (2.1) and (2.3) uniquely determine $V_L(q)$. The difficulty is to prove consistency, *i.e.* that $V_L(q)$ depends on the link L (up to isotopy) and not on any particular plane projection.

Note.- In fact $V_L(q)$ does not depend on the orientation of L . However this is not true for the generalizations of $V_L(q)$, except that reversing the orientation of *all* components of L will always preserve the generalized Jones polynomials.

Example.- For a (right-handed) trefoil knot $V(q) = -q^4 + q^3 + q$. By (2.2) this distinguishes it from its mirror image, the left-handed trefoil.

Although it is possible to verify the consistency of (2.3) by direct combinatorial methods this is not very enlightening. A better approach, explained in [3], is based on the use of *braids*.

The Artin braid group on n strands B_n can be defined as the fundamental group of the configuration space C_n of n unordered distinct points in the plane. There is an elementary geometric construction which assigns to any braid β an oriented link $\hat{\beta}$ in S^3 . All links arise in this way and the equivalence relation on the union of all B_n given by

$$\beta_1 \sim \beta_2 \Longleftrightarrow \widehat{\beta_1} \text{ isotopic to } \widehat{\beta_2}$$

is explicitly known. Thus one may construct link invariants from suitable braid invariants.

To get the right braid invariants to produce $V_L(q)$, Jones introduces certain representations $\rho_\lambda(q)$ of B_n .

These are representations depending on a parameter q and a partition λ of n . For $q = 1$ they reduce to the irreducible representations ρ_λ of the symmetric group S_n pulled back to B_n via the natural homomorphism $B_n \to S_n$. The representations $\rho_\lambda(q)$ come from representations of the Hecke algebra.

The Jones polynomial $V_L(q)$ for $L = \hat{\beta}$ is now defined as a certain linear combination of the characters of $\rho_\lambda(q)$ evaluated at β . The only partitions λ which are needed here are the partitions of n into (at most) 2 parts.

Note.- For generalizations of $V_L(q)$ one needs all partitions of n . These generalizations lead to polynomials satisfying suitable generalizations of (2.3).

In this braid group approach to the Jones polynomial it is still a mystery why suitable linear combinations of the characters $\rho_\lambda(q)$ should give link invariants. The underlying reason becomes clear in Witten's theory as we shall see.