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SINGULAR INTEGRALS AND RECTIFIABLE SETS IN $\mathbb{R}^n$

Au-delà des graphes lipschitziens

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Introduction

There are a number of natural ways to look at the goals and results of this monograph. The first can be stated broadly as the problem of relating the geometry of a set E in $\mathbf{R}^n$ to the analysis of functions and linear operators on E. A specific question of this type that we shall be concerned with here is the following. Let E be a subset of $\mathbf{R}^n$ that has Hausdorff dimension d, $0 < d < n$. We equip E with d-dimensional Hausdorff measure restricted to it, and we assume that this measure is locally finite. Under what conditions on E is it true that plenty of singular integral operators are bounded on $L^2(E)$? Examples of the sort of singular integral operators that we have in mind are the Cauchy integral when $d = 1$ and $n = 2$ and the double-layer potential when $d = n - 1$.

It is known from the work of Coifman, McIntosh, and Meyer **[CMM]** that this is true when E is a Lipschitz graph. There are several more general conditions on E which are known to be sufficient to ensure the boundedness of lots of singular integral operators, but there has not been much progress on finding necessary conditions. Our main result provides geometrical characterizations of the sets E for which a fairly large class of singular integral operators are bounded on $L^2(E)$, at least if we make an auxiliary technical assumption on E (Ahlfors regularity). See Section 1 for the precise statement. Unfortunately we do not know at this time how to work with smaller classes of operators; for example, when $d = 1$ and $n = 2$ we would like to use only the boundedness of the Cauchy integral.

The geometrical conditions that arise in the aforementioned theorem can be thought of as quantitative analogues of the classical notion of rectifiability. Recall that E is said to be rectifiable if it is contained in the union of a countable family of Lipschitz images of $\mathbf{R}^d$, except for a set of d-dimensional Hausdorff measure zero. Rectifiability is a qualitative condition, and it is not strong enough to imply the boundedness of singular integral operators. Using real-variable methods as in [**D1**, **3**] it can be shown that various quantitative versions of rectifiability are strong enough to imply the boundedness of plenty of singular integral operators, and our theorem provides a converse to this.

There is a great deal of information available about rectifiable sets. See [**Fe**], [**Fl**], and [**Ma**], for instance. Not so much seems to be known about quantitative analogues of rectifiability. In particular there are many characterizations of rectifiability, and these give rise to many candidates for the notion of quantitative rectifiability, but the complete relationship between these various candidates is not at all clear. Our theorem provides some nontrivial equivalences between some of these conditions. Although this is a purely geometrical issue, it turns out that singular integral operators provide a useful tool for passing between some of these conditions.

Our main result also gives a higher-dimensional version of Peter Jones' travelling salesman theorem ([**J3**]). That is, we give two other conditions on E that are equivalent to the others, and which are roughly as follows. One of these conditions says that E is contained in a set that admits a nice parameterization by $\mathbf{R}^d$. The other condition is a bound on certain quantities that measure the extent to which E can be approximated by d-planes. Again, Section 1 should be consulted for the precise statement.

Although there are several ways of looking at what we are doing and what it means, there is an underlying common theme. To a large degree we are trying to produce methods for analyzing the geometry of sets, in much the same way that more traditional harmonic analysis (as in [**St**]) is concerned with the analysis of functions and operators. Some of the ideas of harmonic analysis make sense in this context, but mostly the techniques don't work so well, because of the absence of a linear structure. The methods that have grown out of Carleson's corona construction seem to be more cooperative in this geometrical setting.

In connection with the analogy with traditional harmonic analysis it is interesting to look at the theorem in Section 1 from the perspective of Littlewood-Paley theory. In some sense this theorem gives a Littlewood-Paley characterization of a class of good sets that is analogous to well-known results for Sobolev spaces. It turns out that this analogy is somewhat misleading, in that there are some other results in our geometrical context that do not have a natural counterpart for Sobolev spaces. Such a result is discussed just before Lemma 5.13, but its details will appear elsewhere.