RAPHAËL CERF

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Raphaël Cerf

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Raphaël Cerf
CNRS, UMR 8628
Université Paris Sud, Mathématique, Bâtiment 425, 91405 Orsay Cedex, France.
E-mail : `Raphael.Cerf@math.u-psud.fr`
Url : `http://www.math.u-psud.fr/~cerf`

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LARGE DEVIATIONS FOR THREE DIMENSIONAL SUPERCRITICAL PERCOLATION

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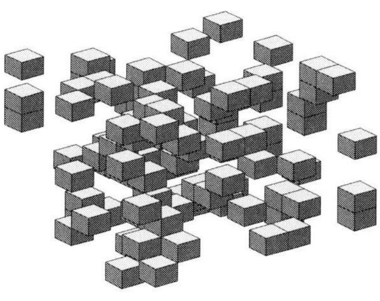
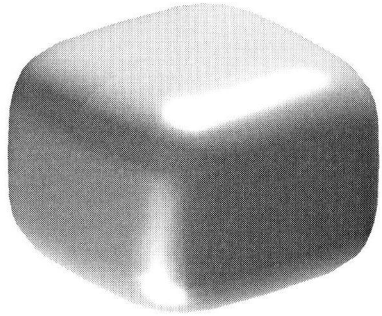
Abstract. We consider Bernoulli bond percolation on the three dimensional cubic lattice in the supercritical regime. We prove a large deviation principle for the rescaled configuration, from which a picture of the Wulff crystal of the model emerges.

Résumé (Grandes déviations pour la percolation supercritique en dimension trois). Nous considérons la percolation Bernoulli sur les arêtes du réseau cubique de dimension trois dans le régime supercritique. Nous prouvons un principe de grande déviation pour la configuration renormalisée, duquel émerge une image du cristal de Wulff du modèle.





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CHAPTER 1

INTRODUCTION

The ground–breaking monograph of Dobrushin, Kotecký and Shlosman [31] has initiated in the past years an intense research activity around the study of phase separation for the two dimensional Ising model (see [21,46,47,48,49,58,59,66,67,68]). The so–called Wulff construction in this context is now fairly well understood. The next challenge is to analyze phase separation and coexistence in higher dimensions. The aim of this work is to propose a way to do that for the three dimensional Bernoulli percolation model.

We consider Bernoulli bond percolation on the three dimensional lattice $\mathbb{Z}^3$. Edges between nearest neighbours are independently open with probability p and closed with probability $1-p$. It is known that this model has a phase transition at a value p_c strictly between 0 and 1: for $p < p_c$ the open clusters are finite and for $p > p_c$ there exists a unique infinite open cluster C_∞ (see [42]). We focus here on the supercritical regime where $p > p_c$. Aside from the infinite cluster, the configuration contains finite clusters of arbitrary large sizes. We wish to understand the geometry of these large clusters. The presence at a particular location of a large finite cluster is an event of low probability: for Bernoulli percolation in dimension d, for $p > p_c$, there exist two positive constants c_1, c_2 such that

$$\forall n \in \mathbb{N} \quad \exp(-c_1 n^{(d-1)/d}) \leq P(n \leq \operatorname{card} C(0) < \infty) \leq \exp(-c_2 n^{(d-1)/d}),$$

where $C(0)$ is the open cluster containing the origin. This is a result from Kesten and Zhang [50] (It was proved under the assumption that p is strictly larger than the limit of the slab critical points. Grimmett and Marstrand proved that this limit coincides with p_c, see [43] or the second edition of [42]). An historic account of the successive refinements of this type of bounds is given in [3]. This estimate is based on the fact that the occurrence of a large finite cluster is due to a surface effect. Indeed at the frontier of a large open cluster C, there is a set $\partial_e C$ of closed edges, called the edge boundary of C, whose macroscopic components look like a large surface separating the sites of the cluster from the outside world.

Alexander, Chayes and Chayes have obtained much more precise results in the two dimensional case [3] (which were further refined by Alexander [2]). Let us sum up the main points of their work. In dimension two, a component of the edge boundary is a closed curve of the plane. The most likely curves to realize the event

$\{ n \leq \operatorname{card} C(0) < \infty \}$ are close to a very specific deterministic curve, namely the boundary of the so–called Wulff crystal of the model. More precisely, it is possible to define an angle dependent surface tension $\sigma(\nu)$ for the model, which characterizes the exponential decay of the probability of having a long flat interface of closed edges orthogonal to the direction ν. Consider next the variational problem of minimizing the σ–surface energy of a closed curve γ (that is the linear integral of $\sigma(\nu)$ along γ, where ν is the normal to γ) under the constraint that the curve γ encloses an area at least one. The unique solution of this problem is the boundary of a suitable dilation of the Wulff crystal of σ

$$\mathcal{W}_\sigma = \{ x \in \mathbb{R}^2 : x \cdot \nu \leq \sigma(\nu) \text{ for all unit vectors } \nu \} .$$

Besides, the probability that the cluster $C(0)$ has inside the macroscopic components of $\partial_{\mathrm{e}} C(0)$ a density larger than the typical density $\theta(p)$ of the infinite cluster is of order $\exp(-\text{const} \operatorname{card} C(0))$, because this event requires a volume effect. Hence, up to surface order large deviation events, the area enclosed by the macroscopic components of $\partial_{\mathrm{e}} C(0)$ and $\theta^{-1} \operatorname{card} C(0)$ are comparable. Finally,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \ln P(n \leq \operatorname{card} C(0) < \infty) = -(\theta \operatorname{area} \partial\mathcal{W}_\sigma)^{-1/2} \int_{\partial\mathcal{W}_\sigma} \sigma(\nu_{\mathcal{W}_\sigma}(x))\, dx .$$

Furthermore the solutions of the previous variational problem are stable with respect to the Hausdorff distance between curves, that is, up to translations, any minimizing sequence converges towards the boundary of a suitable dilated Wulff crystal. As a consequence, conditionally on the event $\{ n \leq \operatorname{card} C(0) < \infty \}$, with probability tending rapidly to 1 as n goes to ∞ (say faster than any inverse power of n), the Hausdorff distance between the rescaled curve $n^{-1/2}$(outer component of $\partial_{\mathrm{e}} C(0)$) and the boundary of the dilated Wulff crystal $(\theta \operatorname{area} \partial\mathcal{W}_\sigma)^{-1/2} \mathcal{W}_\sigma$ is less than any fixed positive real number. Alexander, Chayes and Chayes prove also a single droplet Theorem, which is close in spirit to the Wulff construction Theorem of Dobrushin, Kotecký and Shlosman [31] for the two dimensional Ising model: for $\lambda < (\operatorname{diam} \mathcal{W}_\sigma)^{-2} \operatorname{area} \partial\mathcal{W}_\sigma$, for any positive η, conditionally on the event

$$\{ \operatorname{card} C_\infty \cap [-L/2, L/2]^2 \leq (1-\lambda)\theta L^2 \} ,$$

(C_∞ is the infinite cluster), with probability tending rapidly to one as L goes to ∞, there is inside $[-L/2, L/2]^2$ a finite open cluster of cardinality approximatively $\theta\lambda L^2$ whose edge boundary is at a Hausdorff distance less than ηL from $\lambda^{1/2} L \, (\operatorname{area} \partial\mathcal{W}_\sigma)^{-1/2} \partial\mathcal{W}_\sigma$.

Our original motivation was to prove similar results in the three dimensional case. As noted in [3,31], a new formulation of the results themselves is required in higher dimension: indeed the Hausdorff distance between the boundaries is not adequate any more, because very thin and long filaments create insignificant surface energy while increasing dramatically the Hausdorff distance. However the main obstacle so far seems to have been the extension of the skeleton coarse graining technique [49] which relies on the possibility of approximating in a suitable way a polygonal line on the lattice by a coarser one and on a combinatorial bound for the number of polygonal lines. The structure of two dimensional surfaces is so rich compared to the one of curves that it seems hard to find a similar combinatorial argument in higher dimension (see the introduction of [49]). Therefore a new strategy is needed. A natural way is to leave the discrete setting and to try to work from the start into the continuum. The combinatorial argument should then be replaced by a compactness property. Thus we must embed our objects into a continuous topological space in which the level sets of the surface energy are compact. If the volume happened to be continuous, we would have in addition existence of solutions for our variational problems, a highly desirable feature. This picture has already a taste of large deviations theory: the surface energy should be a good rate function. Subsequently, why not seek for a large deviation principle (in this yet undefined ideal space) governed by the surface energy? The results concerning the Wulff crystal would then be natural consequences of the large deviation principle; the law of the random objects under a volume constraint would concentrate exponentially fast around the ones minimizing the surface energy with respect to this volume constraint. Indeed, whenever a large deviation principle holds, the random objects solve automatically the variational problems associated with the rate function. Thus it is very reasonable to think that the probabilistic results on the Wulff crystal (at least those dealing with rough estimates) might be included in a general large deviations setting. Large deviations theory itself does not provide the required probabilistic tools, yet it suggests efficient abstract guidelines to attack the problem.

Although there exists a substantial literature devoted to the study of stochastic geometry, we found no existing result on large deviations for general random sets. Thus we started by proving the simplest such result, namely the analog of the Cramér Theorem for random sets [17]. We then tested the feasibility of some aspects of the large deviations approach to the Wulff crystal in the case of two dimensional Bernoulli percolation [18]. There we prove large deviation principles for the finite cluster shape in the Hausdorff and L^1 metric, but with the help of the skeleton coarse graining technique, instead of working from the start in the continuum.

To achieve this appealing programme, we should first find the ideal continuous space to work in. It is clear that this space must contain the smooth surfaces Γ and that the surface energy $\mathcal{I}(\Gamma)$ of a smooth surface Γ has to be

$$\mathcal{I}(\Gamma) = \int_\Gamma \tau(\nu_\Gamma(x))\, d\mathcal{H}^2(x)$$

where $\nu_\Gamma(x)$ is the normal vector to Γ at x, $\tau(\nu)$ is the surface tension of the model in the direction ν and $\mathcal{H}^2$ is the two dimensional Hausdorff measure in $\mathbb{R}^3$. The