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NONDEGENERACY OF NONRADIAL SIGN-CHANGING SOLUTIONS TO THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. — We prove that the non-radial sign-changing solutions to the nonlinear Schrödinger equation

$$\Delta u - u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N, \quad u \in H^1\left(\mathbb{R}^N\right)$$

constructed by Musso, Pacard, and Wei [16] are non-degenerate. This provides the first example of a non-degenerate sign-changing solution to the above nonlinear Schrödinger equation with finite energy.

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RÉSUMÉ (*Non dégéneration de solutions non radiales qui changent de signe à l'équation non linéaire de Schrödinger*). — Nous prouvons que les solutions non radiales qui changent de signe à l'équation non linéaire de Schrödinger

$$\Delta u - u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N,\ u \in H^1\left(\mathbb{R}^N\right)$$

qui ont été construits par Musso, Pacard, Wei [16] sont non dégénérées. Ceci fournit le premier exemple de solutions qui changent de signe à l'équation non linéaire de Schrödinger avec énergie finie.

1. Introduction and statement of main results

In this paper, we consider the nonlinear semi-linear elliptic equation

$$\Delta u - u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N), \tag{1}$$

where p satisfies $1 < p < \infty$ if $N = 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. Equation (1) arises in various models in mathematical physics and biology. In particular, the study of standing waves for the nonlinear Klein-Gordon or Schrödinger equations reduces to (1). We refer the reader to the papers of Berestycki and Lions [4, 3], and Bartsch and Willem [2] for further references and motivation.

Denote the set of nonzero finite energy solutions to (1) by

$$\Sigma := \left\{u \in H^1(\mathbb{R}^N) : \Delta u - u + |u|^{p-1}u = 0\right\}.$$

If $u \in \Sigma$ and $u > 0$, then the classical result of Gidas, Ni, and Nirenberg [7] asserts that u is radially symmetric. Indeed, it is known [9, 12] that there exists a unique radially symmetric (in fact radially decreasing) positive solution for

$$\Delta w - w + w^p = 0 \quad \text{in} \quad \mathbb{R}^N,$$

which tends to 0 as $|x|$ tends to ∞. All of the other positive solutions to (1) belonging to Σ are translations of w.

Let L_0 be the linearized operator around w, defined by

$$L_0 := \Delta - 1 + pw^{p-1}. \tag{2}$$

Then, the natural invariance of problem (1) under the group of isometries in $\mathbb{R}^N$ reduces to the fact that the functions

$$\partial_{x_1} w, \ldots, \partial_{x_N} w \tag{3}$$

naturally belong to the kernel of the operator L_0. The solution w is *non-degenerate* in the sense that the L^∞-kernel of the operator L_0 is spanned by the functions given in (3). For further details, see [18].

No other example is known of a *non-degenerate* solution to (1) in Σ. The purpose of this paper is to provide the first example other than w of a *non-degenerate* solution to (1) in Σ.

Concerning the existence of other solutions to (1) in Σ, several results are available in the literature. Berestycki and Lions [4, 3] and Struwe [23] have demonstrated the existence of infinitely many radially symmetric sign-changing solutions. The proofs of these results are based on the use of variational methods. The existence of non-radial sign-changing solutions was first proved by Bartsch and Willem [2] in dimensions $N = 4$ and $N \geq 6$. In this case, the result is also proved by means of variational methods, and the key idea is to look for solutions that are invariant under the action of $O(2) \times O(N-2)$, in order to recover a certain compactness property. The result was subsequently generalized by Lorca and Ubilla [14] to the $N = 5$-dimensional case. Other than the symmetry property of the solutions, these results do not provide any qualitative properties of the solutions. A different approach, and a different construction, have been developed in [16] and [1], where new types of non-radial sign-changing finite-energy solutions to (1) are constructed, and a detailed description of these solutions is provided.

The main purpose of this paper is to prove that the solutions constructed by Musso, Pacard, and Wei in [16] are rigid, up to transformations of the equation. In other words, these solutions are *non-degenerate* in the sense of the definition introduced by Duyckaerts, Kenig, and Merle [6].

To explain the meaning of a *non-degenerate* solution for a given $u \in \Sigma$, we recall all of the possible invariance of equation (1). We have that equation (1) is invariant under the following two transformations:

(1) (translation): If $u \in \Sigma$, then $u(x+a) \in \Sigma$ $\forall a \in \mathbb{R}^N$;
(2) (rotation): If $u \in \Sigma$, then $u(Px) \in \Sigma$, where $P \in O_N$, and O_N is the classical orthogonal group.

If $u \in \Sigma$, then by

$$L_u = \Delta - 1 + p|u|^{p-1} \tag{4}$$

we denote the linearized operator around u. We define the null space of L_u as

$$\mathcal{Z}_u = \{f \in H^1(\mathbb{R}^N) : L_u f = 0\}. \tag{5}$$

If we denote the group of isometries of $H^1(\mathbb{R}^N)$ generated by the previous two transformations by $\mathcal{M}$, then the elements in $\mathcal{Z}_u$ generated by the family of transformations $\mathcal{M}$ define the following vector space:

$$\tilde{\mathcal{Z}}_u = \operatorname{span} \left\{ \begin{array}{l} \partial_{x_j} u, \ 1 \leq j \leq N \\ (x_j \partial_{x_k} - x_k \partial_{x_j}) u, \ 1 \leq j < k \leq N \end{array} \right\}. \tag{6}$$

A solution u to (1) is *non-degenerate* in the sense of Duyckaerts, Kenig, and Merle [6] if

$$\mathcal{Z}_u = \tilde{\mathcal{Z}}_u. \tag{7}$$

As was already mentioned, the only non-degenerate example of $u \in \Sigma$ known so far is the positive solution w. In fact, in this case

$$(x_j \partial_{x_k} - x_k \partial_{x_j}) w = 0, \quad \forall \quad 1 \leq j < k \leq N,$$

and hence

$$\mathcal{Z}_w = \operatorname{span}\left\{\partial_{x_j} w,\ 1 \leq j \leq N\right\}.$$

The proof of the non-degeneracy of w relies heavily on the radial symmetry of w. For non-radial solutions, the strategy used to prove non-degeneracy in the radial case is no longer applicable. Thus, a new strategy is required for non-radially symmetric solutions.

A similar problem has arisen in the study of non-radial sign-changing finite-energy solutions for the Yamabe-type problem

$$\Delta u + |u|^{\frac{4}{N-2}} u = 0 \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

for $N \geq 3$. In [17], the second and third authors of the present paper introduced some new ideas for dealing with non-degeneracy in non-radial sign-changing solutions to the above problem. Indeed, they successfully analyzed the non-degeneracy of some non-radial solutions to the Yamabe problem that were previously constructed in [20]. For other constructions, we refer the reader to [21]. In this paper, we will adopt the idea developed in [17] to analyze the non-degeneracy of the solutions to (1) constructed by Musso, Pacard, and Wei [16].

The main result of this paper can be stated as follows.

Theorem 1.1. — *There exists a sequence of non-radial sign-changing solutions to (1) with arbitrarily large energy, and each solution is non-degenerate in the sense of (7).*

We believe that the non-degeneracy property of the solutions in Theorem 1.1 can be used to obtain new types of constructions for sign-changing solutions to (1), or related problems in bounded domains with Dirichlet or Neumann boundary conditions. We will address this problem in future work.

This paper is organized as follows. In Section 2, we introduce the solutions constructed by Musso, Pacard, and Wei in [16]. In Section 3, we sketch the main steps, and present the proof of Theorem 1.1. Sections 4 to 8 are devoted to the proof of properties required for the proof of Theorem 1.1.

2. Description of the solutions

In this section, we describe the solutions u_ℓ constructed in [16], and recall some properties that will be useful later. To provide the description of the solutions, we introduce some notations. The canonical basis of $\mathbb{R}^N$ will be denoted by

$$\mathbf{e}_1 = (1, 0, \cdots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_N = (0, \cdots, 0, 1). \tag{8}$$

Let k be an integer, and assume that we are given two positive integers m, n and two positive real numbers $\ell, \bar{\ell}$, which are related by

$$2 \sin \frac{\pi}{k} m\ell = (2n-1)\bar{\ell}. \tag{9}$$

We shall comment on the possible choices of these parameters later on. Consider the regular polygon in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$ with k edges whose vertices are given by the orbit of the point

$$y_1 = \frac{\bar{\ell}}{2 \sin \frac{\pi}{k}} \mathbf{e}_1 \in \mathbb{R}^N$$

under the action of the group generated by R_k. Here, $R_k \in O(2) \times O(N-2)$ is the rotation through the angle $\frac{2\pi}{k}$ in the (x_1, x_2) plane. By construction, the edges of this polygon have length $\bar{\ell}$. We refer to this polygon as the *inner polygon*. We define the *outer polygon* as the regular polygon with k edges whose vertices are the orbit of the point

$$y_{m+1} = y_1 + m\ell \mathbf{e}_1$$

under the group generated by R_k. Observe that the distance from y_{m+1} to the origin is given by $m\ell + \frac{\bar{\ell}}{2 \sin \frac{\pi}{k}}$, and thanks to (9), the edges of the *outer* polygon have length $2n\bar{\ell}$.

By construction, the distance between the points y_1 and y_{m+1} is equal to $m\ell$, and by y_j, for $j = 2, \cdots, m$, we denote the evenly distributed points on the segment between these two points. Namely,

$$y_j = y_1 + (j-1)\ell \mathbf{e}_1 \quad \text{for} \quad j = 2, \cdots, m.$$

As mentioned above, the edges of the *outer* polygon have length $2n\bar{\ell}$, and we evenly distribute points y_j, $j = m+2, \cdots, m+2n$, along this segment. More precisely, if we define

$$\mathbf{t} = -\sin \frac{\pi}{k} \mathbf{e}_1 + \cos \frac{\pi}{k} \mathbf{e}_2 \in \mathbb{R}^N,$$

then the points y_j are given by

$$y_j = y_{m+1} + (j-m-1)\bar{\ell}\mathbf{t} \quad \text{for} \quad j = m+2, \cdots, m+2n.$$

We also denote

$$z_h = y_j \text{ for } h = 1, \cdots, 2n-1, \text{ where } h = j-m-1.$$

Let

$$\Pi = \bigcup_{i=0}^{k-1} \left(\{R_k^i y_j \; : \; j = 1, \ldots, m+1\} \right) \cup \left(\{R_k^i z_h : h = 1, \ldots, 2n-1\} \right). \tag{10}$$