Bull. Soc. Math. France
146 (2), 2018, p. 391-426

WEAK MIXING PROPERTIES OF INTERVAL EXCHANGE TRANSFORMATIONS & TRANSLATION FLOWS

by Artur Avila & Martin Leguil

Abstract. — Let $d > 1$. In this paper we show that for an irreducible permutation π which is not a rotation, the set of $[\lambda] \in \mathbb{P}_+^{d-1}$ such that the interval exchange transformation $f([\lambda], \pi)$ is not weakly mixing does not have full Hausdorff dimension. We also obtain an analogous statement for translation flows. In particular, it strengthens the result of almost sure weak mixing proved by G. Forni and the first author in [2]. We adapt here the probabilistic argument developed in their paper in order to get some large deviation results. We then show how the latter can be converted into estimates on the Hausdorff dimension of the set of "bad" parameters in the context of *fast decaying* cocycles, following the strategy of [1].

Résumé (*Propriétés de mélange faible des échanges d'intervalles et des flots de translation*). — Soit $d > 1$. Dans le présent article, nous montrons que pour une permutation irréductible π différente d'une rotation, l'ensemble des paramètres de longueur $[\lambda] \in \mathbb{P}_+^{d-1}$ tels que l'échange d'intervalles $f([\lambda], \pi)$ n'est pas faiblement mélangeant n'a pas dimension de Hausdorff totale. Nous obtenons également un énoncé similaire dans

Texte reçu le 10 mai 2016, modifié le 22 septembre 2016, accepté le 6 février 2017.

Artur Avila, Institut de Mathématiques de Jussieu - Paris Rive Gauche, CNRS UMR 7586, Université Paris Diderot, Sorbonne Paris Cité Sorbonnes Universités, UPMC Université Paris 06, F-75013, Paris, France & IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil. • *E-mail :* `artur@math.univ-paris-diderot.fr`
Martin Leguil, Université Paris Diderot, Sorbonne Paris Cité, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Sorbonne Universités, UPMC Université Paris 06, F-75013 Paris, France & Department of Mathematics, University of Toronto, 40 St George St. Toronto, M5S 2E4, ON, Canada. • *E-mail :* `martin.leguil@utoronto.ca`

Mathematical subject classification (2010). — 37A05, 37A25, 37E35.

Key words and phrases. — Échanges d'intervalles, surfaces de translation, flots de translation, mélange faible, unique ergodicité / Interval exchange transformations, translation surfaces, translation flows, weak mixing, unique ergodicity.

A.A. was partially supported by the ERC Starting Grant Quasiperiodic.

 0037-9484/2018/391/$ 5.00
 doi:10.24033/bsmf.2761

le cas des flots de translation. En particulier, cela améliore le résultat de [2] dans lequel G. Forni et le premier auteur montrent que la propriété de mélange faible est satisfaite pour presque tout paramètre de longueur. Nous adaptons ici l'argument probabiliste développé dans leur article afin de prouver un résultat de grandes déviations. Nous montrons ensuite que ce dernier peut être employé pour obtenir des estimées sur la dimension de Hausdorff de l'ensemble des « mauvais » paramètres dans le contexte des cocycles à *décroissance rapide*, en suivant la stratégie mise au point dans [1].

Introduction

An *interval exchange transformation*, or i.e.t., is a piecewise order-preserving bijection f of an interval I on the real axis. More precisely, I splits into a finite number of subintervals $(I_i)_{i=1,\dots,d}$, $d > 1$, such that the restriction of f to each of them is a translation. The map f is completely described by a pair $(\lambda, \pi) \in \mathbb{R}_+^d \times \mathfrak{S}_d$: λ is a vector whose coordinates $\lambda_i := |I_i|$ correspond to the lengths of the subintervals, and π a combinatorial data which prescribes in which way the different subintervals are reordered after application of f. We will write $f = f(\lambda, \pi)$. In the following, we will mostly consider *irreducible* permutations π, which we denote by $\pi \in \mathfrak{S}_d^0$; this somehow expresses that the dynamics is "indecomposable". Since dilations on λ do not change the dynamics of the i.e.t., we will also sometimes use the notation $f([\lambda], \pi)$, with $[\lambda] \in \mathbb{P}_+^{d-1}$. For more details on interval exchange transformations we refer to Section 3.

A *translation surface* is a pair (S, ω) where S is a surface and ω some nonzero Abelian differential defined on it. Denote by $\Sigma \subset S$ the set of zeros of ω; its complement $S \backslash \Sigma$ admits an atlas such that transition maps between two charts are just translations (see Subsection 3.2 for more details on translation surfaces). Interval exchange transformations can be seen as a discrete version of the geodesic flow on some translation surface, also called a *translation flow*. The introduction of these objects was motivated by the study of the billiard flow on rational polygons, i.e., whose angles are commensurate to π; the relation between these problems is given by a construction called *unfolding*, which associates a translation surface to such a polygon (see for instance [16]) and makes the billiard flow into some translation flow.

In this paper, we are interested in the ergodic properties of interval exchange transformations on $d > 1$ subintervals. It is clear that such transformations preserve the Lebesgue measure. In fact this is often the unique invariant measure: Masur in [9] and Veech in [13] have shown that if the permutation π is irreducible, then for Lebesgue-almost every $[\lambda] \in \mathbb{P}_+^{d-1}$, the i.e.t. $f([\lambda], \pi)$ is uniquely ergodic. As a by-product of our methods, we will see here that in fact, the set of $[\lambda]$ such that $f([\lambda],\pi)$ is not uniquely ergodic does not have full Hausdorff dimension.(1)

In another direction, Katok has proved that i.e.t.s and suspension flows over i.e.t.s with roof function of bounded variation are never mixing with respect to Lebesgue measure, see [4]. Basically, what mixing expresses is that the position of a point at time n is almost independent of its initial position when $n \geq 0$ is large. Let us recall that a measure-preserving transformation f of a probability space (X,m) is said to be *weakly mixing* if for every pair of measurable sets $A,B \in X$, there exists a subset $J(A,B) \subset \mathbb{N}$ of density zero such that

$$\lim_{J(A,B)\not\ni n \to +\infty} m(f^{-n}(A) \cap B) = m(A)m(B). \tag{1}$$

It follows from this definition that every mixing transformation is weakly mixing, and every weakly mixing transformation is ergodic.(2)

From the previous discussion, it is therefore natural to ask whether a typical i.e.t. is weakly mixing or not; this point is more delicate except in the case where the permutation π associated to the i.e.t. $f(\lambda,\pi)$ is a *rotation* of $\{1,\dots,d\}$, i.e., $\pi(i+1) \equiv \pi(i)+1 \mod d$, for all $i \in \{1,\dots,d\}$. Indeed, in this case, the i.e.t. $f(\lambda,\pi)$ is conjugate to a rotation of the circle, hence it is not weakly mixing, for every $\lambda \in \mathbb{R}^d_+$.

It is a classical fact that any invertible measure-preserving transformation f is weakly mixing if and only if it has *continuous spectrum*, that is, the only eigenvalue of f is 1 and the only eigenfunctions are constants. To prove weak mixing we thus rule out the existence of non-constant measurable eigenfunctions.

Let us recall some previous advances in the problem of the prevalence of weak mixing among interval exchange transformations. Partial results in this direction had been obtained by Katok and Stepin [5], who proved weak mixing for almost all i.e.t.s on 3 intervals. In [14], Veech has shown that weak mixing holds for infinitely many irreducible permutations.

The question of almost sure weak mixing for i.e.t.s was first fully answered by Forni and the first author in [2], where the following result is proved:

THEOREM (Theorem A, Avila-Forni [2]). — *Let π be an irreducible permutation of $\{1,\dots,d\}$ which is not a rotation. For Lebesgue almost every $\lambda \in \mathbb{R}^d_+$, the i.e.t. $f(\lambda,\pi)$ is weakly mixing.*

They also obtain an analogous statement for translation flows.

1. It was pointed out to us by J. Athreya and J. Chaika that more is true actually: the results of Masur [10] imply that the Hausdorff codimension of $[\lambda] \in \mathbb{P}^{d-1}_+$ such that $f([\lambda],\pi)$ is not uniquely ergodic is at least 1/2.

2. Indeed, mixing holds when we can take $J(A,B) = \emptyset$ in (1). For ergodicity, if the set A is f-invariant, the choice $B = A$ in (1) yields that A has either full or zero measure.

Theorem (Theorem B, Avila-Forni [2]). — *For almost every translation surface (S,ω) in a given stratum of the moduli space of translation surfaces of genus $g>1$, the translation flow on (S,ω) is weakly mixing in almost every direction.*

Although translation flows can be seen as suspension flows over i.e.t.s, the second result is not a direct consequence of the first one since the property of weak mixing is not invariant under suspensions and time changes.

Note that the previous results tell nothing about zero measure subsets of the moduli space of translation surfaces; in particular, the question of weak mixing was still open for translation flows on Veech surfaces, which are exceptionally symmetric translation surfaces associated to the dynamics of rational polygonal billiards. This problem was solved in [1] by Delecroix and the first author:

Theorem (Theorem 2, Avila-Delecroix [1]). — *The geodesic flow in a non-arithmetic Veech surface is weakly mixing in almost every direction. Indeed, the set of exceptional directions has Hausdorff dimension less than one.*

The proof of almost sure weak mixing in [2] is based on some parameter exclusion; however this reasoning is not adapted to the particular case of translation flows on Veech surfaces. In [1], the authors have developed another strategy to deal with the problem of prevalent weak mixing, based on considerations on Hausdorff dimension and its link to some property that we refer to as *fast decay* in what follows (see Subsection 2.4 for the definition).

In the present paper, we improve the "almost sure" statement obtained in [2]; the proof we give owes much to the ideas developed in [2] and [1]. Our main result is the following.

Theorem A. — *Let $d>1$ and let $\pi\in\mathfrak{S}_d^0$ be an irreducible permutation which is not a rotation; then the set of $[\lambda]\in\mathbb{P}_+^{d-1}$ such that $f([\lambda],\pi)$ is not weakly mixing has Hausdorff dimension strictly less than $d-1$.*

We also get a similar statement for translation flows. Let $d>1$, and $\pi\in\mathfrak{S}_d^0$ which is not a rotation; we consider the translation flows which are parametrized by a pair $(h,[\lambda])\in H(\pi)\times\mathbb{P}_+^{d-1}$, where $\dim(H(\pi))=2g$ (we refer to Subsection 3.3 for a definition). By [2] we know that the set of $(h,[\lambda])\in H(\pi)\times\mathbb{P}_+^{d-1}$ such that the associate flow is weakly mixing has full measure. Here we obtain

Theorem B. — *The set of $(h,[\lambda])\in H(\pi)\times\mathbb{P}_+^{d-1}$ such that the associate translation flow is not weakly mixing has Hausdorff dimension strictly less than $2g+d-1$.*

Acknowledgements

We would like to thank Sébastien Gouëzel and the referee for their very careful reading of this paper and many useful comments. The second author is also grateful to Julie Déserti and Harold Rosenberg for their constant support, and thanks *École polytechnique* and *Réseau Franco-Brésilien de Mathématiques* for their financial support during several visits at Instituto de Matemática Pura e Aplicada in Rio de Janeiro.

1. Outline

The property of weak mixing we are interested in concerns the dynamics of some i.e.t. f on the interval I, or in other terms, *phase space.* But f is parametrized by $([\lambda], \pi) \in \mathbb{P}_+^{d-1} \times \mathfrak{S}_d^0$ and we will see that it is possible to define a dynamics on the *space of parameters* as well. The so-called *Veech criterion* gives a link between the property of weak mixing for phase space and the dynamics of some cocycle in parameter space.

Indeed, "bad" parameters, i.e., corresponding to i.e.t.s which are not weakly mixing, can be detected through a cocycle (T, A) derived from the Rauzy cocycle. For a parameter $[\lambda]$, the weak-stable lamination $W^s([\lambda])$ is defined to be the set of vectors h whose iterates under the cocycle get closer and closer to the lattice $\mathbb{Z}^d$. Veech criterion tells us that the weak-stable lamination $W^s([\lambda])$ associated to some "bad" parameter $[\lambda]$ contains the element $(t, \dots, t)$ for some $t \in \mathbb{R}\backslash\mathbb{Z}$. Prevalent weak mixing can thus be obtained by ruling out intersections between $\mathrm{Span}(1, \dots, 1)\backslash\mathbb{Z}^d$ and $W^s([\lambda])$ for typical $[\lambda]$. In [2], typical meant "almost everywhere"; following the strategy developed by Delecroix and the first author in [1], we show here that this holds actually for every $[\lambda]$ but for a set whose Hausdorff dimension is not maximal, which is stronger.

The study of the weak-stable lamination of (T, A) was done in [2]. To achieve this, and given $\delta > 0$, $m \in \mathbb{N}$, the authors introduce the set $W_{\delta,m}([\lambda])$ of vectors h with norm less than δ and such that the iterates $A_k([\lambda]) \cdot h$ remain small for the pseudo-norm $\| \cdot \|_{\mathbb{R}^d/\mathbb{Z}^d}$ up to time m.

The analysis is based on the following process: given a little segment J near the origin, its image by the cocycle may again contain a point near some element $c \in \mathbb{Z}^d$. When $c = 0$, the corresponding line is called a *trivial child* of J. Else, we translate the image of J by $-c$ to bring it back to the origin, and we call this new segment a *non-trivial child* of J. Note that there may be several non-trivial children. The goal of the study is to show that for most $[\lambda]$, this process has finite life expectancy, that is, the family generated by a line is finite.

A key ingredient in the "local" analysis near the origin is the existence of two positive Lyapunov exponents for surfaces of genus at least 2. For a fixed segment not passing through the origin, the biggest Lyapunov exponent is responsible for the growth of the length of its iterates by the cocycle, while the second biggest generates a drift that tends to kick them further away from the origin.