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SEMICLASSICAL ANALYSIS FOR HARPER'S EQUATION III
CANTOR STRUCTURE OF THE SPECTRUM

by

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RESUME. Dans ce travail nous continuons notre étude de l'opérateur de Harper, $\cos hD+\cos x$ dans $L^2(\mathbb{R})$, par des méthodes d'analyse microlocale et de renormalisation. On obtient une description assez complète du spectre dans le cas où $h/2\pi$ est irrationnel avec un développement en fraction continue : $h/2\pi=1/(q_0+1/(q_1+\dots))$, si $q_j\in\mathbb{Z}$, $|q_j|\geq C_0$ et $C_0>0$ est assez grand. En particulier le spectre est un ensemble de Cantor de mesure 0. Nos résultats sont aussi valables pour certaines perturbations de l'opérateur de Harper et on donne une application à l'opérateur de Schrödinger magnétique périodique sur $\mathbb{R}^2$.

ABSTRACT. In this paper we continue our study of Harper's operator $\cos hD+\cos x$ in $L^2(\mathbb{R})$, by means of microlocal analysis and renormalization. A rather complete description of the spectrum is obtained in the case when $h/2\pi$ is irrationnel and has a continued function expansion : $h/2\pi=1/(q_0+1/(q_1+\dots))$ with $q_j\in\mathbb{Z}$, $|q_j|\geq C_0$, provided that $C_0>0$ is sufficiently large. In particular, the spectrum is a Cantor set of measure 0. Our results are also valid for certain perturbations of Harper's operator and an application to the periodic magnetic Schrödinger operator on $\mathbb{R}^2$ is given.

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0. Introduction.

This work is a continuation of our study, started in [HS1,2], of the spectrum of Harper's operator, by the use of semi-classical methods. If $h \in \mathbb{R}$, $h \neq 0$, then the problem is to study the union of the spectra, when θ varies in $\mathbb{R}$, of the operators in $\mathcal{L}(l^2(\mathbb{Z}), l^2(\mathbb{Z}))$, given by,

$$H_\theta u(n) = \tfrac{1}{2}(u(n+1) + u(n-1)) + \cos(hn+\theta)u(n). \tag{0.1}$$

As a set, this union of spectra coincides with the spectrum of,

$$P_0 = \cos(hD_x) + \cos(x) \tag{0.2}$$

in $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$, where $D_x = i^{-1}\partial/\partial x$, so that $\cos(hD_x) = \frac{1}{2}(\tau_h + \tau_{-h})$, where $\tau_h u(x) = u(x-h)$. Inspired by ideas of Wilkinson [W1], we obtained a partial Cantor structure result for the spectrum, $Sp(P_0)$ of P_0 under the assumption that $h/2\pi$ is irrational and,

$$h/2\pi = 1/(q_1 + 1/(q_2 + \ldots)), \quad q_j \in \mathbb{Z}, \ 1 \leq j < \infty, \tag{0.3}$$

and

$$|q_j| \geq C_0, \tag{0.4}$$

for some sufficiently large constant C_0. (See Théorème 1 in [HS1].) Roughly, our result was that if $\varepsilon_0>0$ and if $C_0>0$ is sufficiently large (as a function of ε_0), then outside $[-\varepsilon_0,\varepsilon_0]$ the spectrum of P_0 is contained in a union of intervals, J_j of width $\exp(-\sim|q_1|)$ and such that the separation between neighboring intervals (on the same side of $[-\varepsilon_0,\varepsilon_0]$) is $\sim 1/|q_1|$. If κ_j is the increasing affine function that maps J_j onto $[-2,2]$, then outside $[-\varepsilon_0,\varepsilon_0]$, the set $\kappa_j(J_j\cap Sp(P_0))$ can again be localized into a finite union of closed intervals, having widths and separations of the same order of magnitude as for the J_j , but in terms of q_2 instead of q_1. This procedure can then be continued indefinitely.

The proof of this result was obtained by applying first microlocal analysis near a "potential well", i.e. a component of $\cos(\xi)+\cos(x)=\mu$, where $\mu\in[-2,2]\setminus[-\varepsilon_0,\varepsilon_0]$, in order to obtain certain discrete eigenvalues, well defined up to $\mathcal{O}(e^{-1/Ch})$. It then followed that $Sp(P_0)$ is localized to certain intervals, J_j, exponentially close to these eigenvalues. After that we analyzed the tunnel effect between the potential wells, and this permitted us to describe $Sp(P_0)\cap J_j$ as the spectrum of a certain infinite "interaction" matrix. Exploiting certain translation invariance properties of the resulting matrix, we could then reduce the study of it's spectrum to that of $P(x,h'D_x)$, the Weyl quantization of $P(x,h'\xi)$, (and by definition, the h'-Weyl quantization of $P(x,\xi)$,) where $P=P_{j,h}$ is a small perturbation of $P_0=\cos(\xi)+\cos(x)$. Here $h'/2\pi=1/(q_2+1/(q_3+\dots))$. For P we could then start over again ....

In this paper, we shall be able to eliminate the intervals, $[-\varepsilon_0,\varepsilon_0]$, and obtain a fairly complete description of $Sp(P_0)$, under the assumption (0.3),(0.4) with $C_0>0$ sufficiently large. When trying to make this improvement at the first level of the iteration scheme, an obvious difficulty is that for $\mu\approx 0$, $P_0^{-1}(\mu)$ is close to the union of the lines $\xi=\pm x+(2k+1)\pi$, $k\in\mathbb{Z}$, and there is no more obvious localization into potential wells.

As before, we can however study microlocal solutions of the homogeneous equation, $(P_0-\mu)u=0$, and as a matter of fact, this was done heuristically already by Azbel [Az]. Away from the saddle points, $(k\pi,l\pi)$, $k+l\in 2\mathbb{Z}+1$, the characteristic set, $P_0^{-1}(\mu)$ is a smooth analytic curve, and near a point in this part of the set, the microlocal kernel of $(P_0-\mu)$ is a one dimensional space, generated by a standard WKB solution. Near a saddle point the space of microlocal solutions is two dimensional, and can be computed more or less explicitly. If we choose for instance the point $(0,\pi)$, then an element of the microlocal kernel near that point, is determined by its behaviour near the open segments $](-\pi,2\pi),(0,\pi)[$ and $](0,\pi),(\pi,0)[$. Using a microlocal study of $P_0-\mu$, we can then obtain a globally defined, well posed "Grushin" problem,

(0.5) $(P_0-\mu)u+R_-u^-=v,\ R_+u=v^+,$

where, $u,v\in L^2(\mathbb{R})$, $u^-,v^+\in l^2(\mathbb{Z}^2;\mathbb{C}^2)$. Roughly (thinking of the case, $v=0$), the condition $R_+u=v^+$ fixes the microlocal behaviour of u near all segments of the form $]((k-1)\pi,(l+1)\pi),(k\pi,l\pi)[$, $k+l\in 2\mathbb{Z}+1$, and R_-u^- provides a one-dimensional inhomogeneity near each segment of the form, $](k\pi,l\pi),((k+1)\pi,(l+1)\pi)[$. Denoting the solution by,

(0.6) $u=Ev+E_+v^+,\ u^-=E_-v+E_{-+}v_+,$

where all operators depend on μ, it is easy to show that μ belongs to the spectrum of P_0 if and only if 0 belongs to the spectrum of E_{-+}. Now E_{-+} may be viewed as a block matrix, $(E_{-+}(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}^2}$, where each entry is a 2×2 matrix. By the same procedure as for the matrix W above, we then see that $0\in Sp(E_{-+})$ iff $0\in Sp(P)$, where P is a 2×2 matrix of h'-pseudodifferential operators. After rescaling, we see that (in the most interesting spectral region) P falls into a certain class of "strong type 2 operators". We also define strong type 1 operators, as scalar h-pseudodifferential operators, satisfying certain commutation relations and which are close to $P_0(x,hD)$. Fortunately, the study of strong type 2 operators is often very close to the study of s-type 1 operators, and we can again divide the problem into certain potential well cases and a branching case.

An interesting feature is that we loose the linear dependence of the spectral parameter, already after considering the first branching problem, so we shall systematically work with operators $P=P_\mu$, and define μ-Sp(P) as the set of μ such that 0 belongs to the spectrum of P_μ. Theorem 6.2 below shows that the study of the μ-spectrum of a strong type 1 h-pseudodifferential operator sufficiently close to P_0 can, when h is sufficiently small, be localized into into a union of closed disjoint intervals, such that the further study of the μ-spectrum in each of these intervals leads to an operator either of s-type 1 or 2. Theorem 9.2 gives the corresponding result for s-type 2 operators. Theorem 9.3 is a combination of the Theorems

6.2 and 9.2 and says that if we start with a strong type 1 operator sufficiently close to P_0 and if (0.3),(0.4) hold with C_0 sufficiently large, then we get a complete description of the μ-Spectrum by means of an infinite sequence of localizations into finite disjoint unions of closed subintervals and rescalings. From the additional quantitative informations stated after the Theorems 6.2 and 9.2 about the lengths and separations of the various intervals appearing in those theorems, combined with Theorem 9.3, we obtain the following result (which expresses only a small part of the very precise information that our methods produce).

Theorem 0.1. Let $P=P(x,hD_x)$ be a self-adjoint h-pseudodifferential operator such that the corresponding Weyl symbol, $P(x,\xi)$ extends holomorphically to the "band" $|\mathrm{Im}(x,\xi)|<1/\varepsilon$, and satisfies:

(0.7) $P((x,\xi)+2\pi\alpha)=P(x,\xi)$, for all $\alpha\in\mathbb{Z}^2$,

(0.8) $P(\xi,-x)=P(\xi,x)$,

(0.9) $|P(x,\xi)-(\cos(\xi)+\cos(x))|\leq\varepsilon$, when $|\mathrm{Im}(x,\xi)|<1/\varepsilon$.

If (0.3),(0.4) hold with $C_0>0$ sufficiently large, and if $0<\varepsilon<\varepsilon_1$ with $\varepsilon_1>0$ sufficienly small, then Sp(P) is of Lebesgue measure 0, has no isolated points and is nowhere dense. (The last statement means that Sp(P) is dense in no non-trivial open interval.)

As already mentioned, the method produces a much more precise description of the spectrum, which is unfortunately rather lengthy to formulate in terms of a theorem, but the interested reader will be able to extract that information from the proofs. This refined description will no doubt be useful when studying the Hausdorff dimension of the spectrum as a function of the sequence (q_j). From the point of view of applications, it is important that our results apply also to small perturbations of Harper's operator. In appendix e we show that under suitable assumptions, the spectrum of a periodic magnetic Schrödinger operator is near the bottom a Cantor set of measure 0.

In [HS2], the results of [HS1] where extended to the case when for some N: $|q_j|\geq C_N(q_1,..,q_N,\varepsilon_0)$ for $j\geq N+1$, but still with the same incompleteness as in [HS1]. We believe that the techniques of the present paper rather automatically lead to a more complete Cantor structure result also in that case. One could probably generalize the result even to the case when there is a sequence $1\leq j_1<j_2<j_3<\dots$ of integers such that

$|q_{j_k}|\geq C_{j_k-j_{k-1}}(q_{j_{k-1}+1},\dots,q_{j_k-1})$, for suitable functions, C_N.

The plan of the paper is the following:

Section 1. Here we introduce and study certain auxiliary operators.

Section 2 contains a formal study of the iteration steps that we will encounter, and we show that certain crucial symmetries are conserved.

Section 3. Here we treat the potential well problem for s-type 1 operators by suitable modifications of the methods in [HS1].

Section 4 treats the branching problem for strong type 1 operators and we obtain a "renormalized" 2×2 system of h'-pseudodifferential operators.
Sections 5 and 6 contain some preliminary results for the renormalized operator in section 4. It is showed that after rescaling and depending on the spectral region, the renormalized operator is either of s-type 1 or 2. Theorem 6.2 gives the main iteration step for s-type 1 operators.
Section 7 treats the totally degenerate potential well case, which is the only case genuinely non-scalar case for s-type 2 operators. Here we use some ideas from [HS2].
Section 8 treats the non-degenerate potential well case for s-type 2 operators.
Section 9 is devoted to the branching case for s-type 2 operators. Theorem 9.2 gives the main iteration step for s-type 2 operators.

Various results are collected into 5 appendices:
Appendix a contains various general results in microlocal analysis. The paragraph a.1 recollects the approach of [S1] to analytic microlocal analysis via FBI-transforms. We refer to that book for a more thorough treatment. In paragraph a.2 we developp a simple functional calculus for analytic pseudodifferential operators. Paragraph a.3 may be of independent interest. It gives a refined correspondence between unitary Fourier integral operators and canonical transformations.
Appendix b. Here we give local normal forms for self-adjoint pseudodifferential operators when the symbol has a saddle point or a minimum. We only allow unitary conjugations and taking functions of the operator. We believe that the results of this appendix will be useful in other contexts.
Appendix c. Here we show that certain 2×2 systems of pseudodifferential operators can be reduced to the case when the diagonal terms are scalars. This is of use in section 7. See also [HS2].
Appendix d contains some justifications of the arguments in section 4.
Appendix e gives an application to magnetic Schrödinger operators. This is a modification of the corresponding arguments in [HS1]. Since the symmetry (0.8) was never assumed in [HS1], we have to add an extra symmetry to the magnetic and electric fields and check that this leads to (0.8).

Some of the results of the present paper have been announced in [HS3]

We would finally like to thank A.Grigis for a large number of interesting and stimulating dicussions with the authors during the preparation of this long work.

1. Various operators with commutation relations.

In this section, we introduce various auxiliary operators, that will play an important role later, and we study their commutation relations. Some of this was already done in [HS1,2], but we think it is convenient to have all at the same place. Let $h\in\mathbb{R}$, $h\neq 0$. All operators will act on $L^2(\mathbb{R})$. The first operators we study are natural h-quantizations of the translations: $(x,\xi)\rightarrow(x,\xi)+2\pi\alpha$, $\alpha\in\mathbb{Z}^2$ (and sometimes even in $\mathbb{R}^2$). Let $\tau=\tau_{2\pi}$ denote the operator of translation by 2π; $\tau u(x)=u(x-2\pi)$, let τ^* denote the operator of multiplication by $e^{2\pi ix/h}$, and put

(1.1) $T_\alpha=\tau^{\alpha_1}\circ\tau^{*\alpha_2}$, for $\alpha\in\mathbb{Z}^2$.

Sometimes, we shall also use that there is a natural extension of the definition of T_α to the case when $\alpha\in\mathbb{R}^2$, since there is an obvious definition of real powers of τ and τ^*. In a way, the crucial phenomenon that causes all the interesting phenomena for Harper's operator, is that τ and τ^* do not commute in general. In fact, $\tau\circ\tau^*=\exp(-i(2\pi)^2/h)\tau^*\circ\tau$. Let $h'\in\mathbb{R}$, be a number such that,

(1.2) $2\pi/h=k+h'/2\pi$,

for some integer k. Then $\tau\circ\tau^*=\exp(-ih')\,\tau^*\circ\tau$, and more generally we get,

(1.3) $T_\alpha T_\beta= e^{ih'\alpha_2\beta_1}\,T_{\alpha+\beta}$,

(1.4) $T_\alpha T_\beta= e^{ih'\sigma(\alpha,\beta)}T_\beta T_\alpha$,

for $\alpha,\beta\in\mathbb{Z}^2$, where σ denotes the standard symplectic form on $\mathbb{R}^2$, given by $\sigma(x,\xi;y,\eta)=\xi y-x\eta$. (1.3) and (1.4) remain valid for $\alpha,\beta\in\mathbb{R}^2$, provided that we replace h' by $(2\pi)^2/h$. The next operator we introduce is the unitary Fourier transform $\mathcal{F}_h=\mathcal{F}$, which can be viewed as an h-quantization of the map κ: $\mathbb{R}^2\rightarrow\mathbb{R}^2$, given by,

(1.5) $\kappa(x,\xi)=(\xi,-x)$.

Later on we shall also need the maps κ_t: $\mathbb{R}^2\rightarrow\mathbb{R}^2$, defined as rotation by the angle t, so that $\kappa=\kappa_{-\pi/2}$. By definition,

(1.6) $\mathcal{F}u(\xi)=(2\pi h)^{-1/2}\int e^{-ix\xi/h}\,u(x)\,dx$, $h>0$,

and as already mentionned, $\mathcal{F}$: $L^2(\mathbb{R})\rightarrow L^2(\mathbb{R})$ is unitary, $\mathcal{F}^{-1}=\mathcal{F}^*$, where $\mathcal{F}^*$ denotes the complex adjoint in the L^2-sense. It is easy to check, (starting with the operators τ and τ^*,) that,

(1.7) $\mathcal{F}\circ T_\alpha=e^{-ih'\alpha_1\alpha_2}\,T_{\kappa(\alpha)}\circ\mathcal{F}$,

for $\alpha\in\mathbb{Z}^2$, and the same relation with h' replaced by $(2\pi)^2/h$, when $\alpha\in\mathbb{R}^2$.

It will be useful in the following, to recall the relation between $\mathcal{F}$ and the unitary group associated to the harmonic oscillator, $R=\frac{1}{2}(h^2D^2+x^2-h)$, $D=i^{-1}(d/dx)$. Let $U_t=e^{itR/h}$. Since $u_0=e^{-x^2/2h}$ is in the kernel of R, we have $U_tu_0=u_0$. On the other hand, we know (Leray [L]), that $U_{-\pi/2}$ and $\mathcal{F}$ are metaplectic (unitary) operators with the same canonical transformation; $\kappa_{-\pi/2}$, and hence that $U_{-\pi/2}=\omega\mathcal{F}$, for some ω of modulus 1. Since

$U_{-\pi/2}u_0=u_0$ and $\mathcal{F}u_0=u_0$, it follows that $\omega=1$;

(1.8) $U_{-\pi/2}=\mathcal{F}=\mathcal{F}_h$, $h>0$.

For later reference, it will also be convenient to know $U_{\pi/4}$ explicitely. Using that the phase $\varphi(x,y)=-x^2/2+(2)^{1/2}xy-y^2/2$ generates the correct canonical transformation, namely $\kappa_{\pi/4}$, we first see that

(1.9) $U_{\pi/4}u(x)=Ch^{-1/2}\int e^{i\varphi(x,y)/h}u(y)\,dy$,

for some constant C. In order to determine this constant, we again use that $U_{\pi/4}u_0=u_0$ and that the corresponding integral in (1.8) can be evaluated exactly, to obtain that, $C=2^{-1/4}\pi^{-1/2}e^{i\pi/8}$.

Let Γ denote the antilinear operator of complex conjugation; $\Gamma u=\bar{u}$. To this operator we associate the transformation of phase-space; $(x,\xi)\to(x,-\xi)$. We notice that this transformation is anti-canonical, in the sense that the Jacobian is equal to -1. As a general rule we shall associate anti-canonical transformations to antilinear operators. The present association is justified by the following fact. Let $A=A(x,hD)$ be the h-Weyl quantization of the symbol $A(x,\xi)\in S^0(\mathbb{R}^2)=\{a\in C^\infty(\mathbb{R}^2)$; for all $j,k\in\mathbb{N}$, there exists $C_{j,k}$ such that $|\partial_x^j\partial_\xi^k a(x,\xi)|\le C_{j,k}$, for all $(x,\xi)\in\mathbb{R}^2\}$, defined by,

(1.10) $Au(x)=(2\pi h)^{-1}\iint e^{i(x-y)\theta/h}A((x+y)/2,\theta)u(y)\,dy d\theta$, $h>0$,

so that A is $\mathcal{O}(1)$ as a bounded operator on $L^2(\mathbb{R})$ by standard theorems. (See for instance [HS1] for a non standard proof.) When we want to distinguish more clearly between the operator and its symbol, we shall sometimes write $Op_h(A)$ or simply $Op(A)$ for the operator. The justification of the association is then given by,

(1.11) $\Gamma Op(A)=Op(B)\Gamma$,

where $B(x,\xi)=\bar{A}(x,-\xi)$. Notice that $\Gamma^2=id$, so that (1.11) may take many equivalent forms.

Thus in a way, Γ is a natural quantization of reflection in the x-axis. We shall also need quantizations of other reflections, such as in some of the lines $l_\theta=\{t(\cos\theta,\sin\theta);\ t\in\mathbb{R}\}$. To define such reflections, it is natural to rotate l_θ by $\kappa_{-\theta}$ to the x-axis (l_0), then reflect in the x-axis, and finally rotate back again. More precisely, the quantization of reflection γ_θ in l_θ, is defined by,

(1.12) $\Gamma_\theta=U_\theta\Gamma U_{-\theta}$,

so that $\Gamma_0=\Gamma$. This corresponds to $\gamma_\theta=\kappa_\theta\gamma_0\kappa_{-\theta}$. From the definition of U_θ, it is easy to verify that,

(1.13) $\Gamma U_\theta=U_{-\theta}\Gamma$ (,and classically, $\gamma_0\kappa_\theta=\kappa_{-\theta}\gamma_0$),

which gives rise to several obvious equivalent forms of (1.12). We get the general relations,

(1.14) $\Gamma_b U_a=U_\alpha\Gamma_\beta$, $\gamma_b\kappa_a=\kappa_\alpha\gamma_\beta$, if $2b-a=\alpha+2\beta$.

Now it is a general fact, that

(1.15) $U_{-\theta}Op(a)U_\theta=Op(a\circ\kappa_\theta)$,