MÉMOIRES DE LA S. M. F.

ENRIQUE ARRONDO
IGNACIO SOLS

On congruences of lines in the projective space (Chapter 6 written in collaboration with M. Pedreira)

Mémoires de la S. M. F. 2e série, tome 50 (1992)
<http://www.numdam.org/item?id=MSMF_1992_2_50__1_0>

Société Mathématique de France
Mémoire n° 50
Supplément au Bulletin de la S.M.F.
Tome 120, 1992, fascicule 3

ON CONGRUENCES OF LINES IN THE PROJECTIVE SPACE

by Enrique Arrondo and Ignacio Sols
(Chapter 6 written in collaboration with M. Pedreira)

RÉSUMÉ: Nous étudions les congruences lisses (c.a.d. les surfaces de la Grassmannienne $Gr(1,3)$ de droites de $\mathbb{P}^3$) en montrant leur parallélisme avec les surfaces de $\mathbb{P}^4$. Après la description de toutes les congruences lisses de degré au plus neuf et l'étude de son schéma de Hilbert nous développons une théorie générale. Par exemple, nous definissons la notion de liaison adéquate aux congruences et classifions les congruences lisses qui sont projetées de $Gr(1,4)$. Nous trouvons aussi des majorations du genre sectionnel que nous utilisons pour obtenir des conditions (telles que d'avoir une caractéristique d'Euler-Poincaré donnée) qui ne sont vérifiées que par les congruences lisses d'un nombre fini de familles.

SUMMARY: We study smooth congruences (i.e., surfaces in the Grassmannian $Gr(1,3)$ of lines in $\mathbb{P}^3$) showing their parallelism with surfaces in $\mathbb{P}^4$. Besides the description of all smooth congruences up to degree nine and studying their Hilbert scheme, we develop a general theory. For example, we define the adequate notion of liaison for congruences and classify the smooth congruences which are projected from $Gr(1,4)$. We also prove some bounds of the sectional genus in order to give conditions (e.g. having a fixed Euler-Poincaré characteristic) such that there are finitely many families of smooth congruences verifying those conditions.

Texte reçu le 21 août 1990, révisé le 17 juillet 1992
E. Arrondo, I.Sols, Departamento de Algebra, Facultad de Matematicas, Universidad Compultense, 28040 Madrid, Espagne

Table des Matières

INTRODUCTION

The present paper contains with almost no change the thesis of the first author, written under the advice of the second author.

The study of congruences of lines, i.e., surfaces in the Grassmann variety $G=Gr(1,3)$ of lines in $\mathbb{P}^3$, goes back to the middle of last century. From that time, most of the classical algebraic geometers, such as Kummer, Reye, Schumacher, Bordiga, Corrado Segre, Castelnuovo, Fano, Jessop, Semple or Roth, have published papers devoted to this topic, classifying, under certain conditions, congruences of fixed degree or fixed order (see §0 for definitions) or just studying a particular congruence.

However, coinciding with the end of the classical algebraic geometry school, this flurry of research on congruences stopped suddenly. Only in the decade of the 80's, and parallel to the development of the theory of surfaces in $\mathbb{P}^4$, a new interest for congruences started again. To our knowledge, the first paper of this second period (although published later than some other ones) is that of Ziv Ran ([37]) where, solving a conjecture proposed by Sols, he generalizes a classical result of Kummer and classifies all irreducible surfaces of order one in any Grassmannian. After this, many authors have published several papers on congruences: Cossec, Goldstein, Gross, Hernández, Papantonopoulou, Verra, etc.

The reason why this resurgence of the theory of congruences coincides in time with the development of the theory of surfaces in $\mathbb{P}^4$ is that both G and $\mathbb{P}^4$ have the same dimension four (one less than the dimension of a natural ambient space for a smooth surface), so that the same kind of results are expected. In fact, any statement for surfaces in $\mathbb{P}^4$ has its analogous for congruences. The converse, however, is not true, since the geometry of congruences is much richer in problems. For example a congruence has a bidegree, instead of the degree of projective surfaces, and points of the congruence have an interpretation as lines in $\mathbb{P}^3$.

The aim of this work is to prove some general statements on smooth congruences (mainly in §3, §5 and §6), most of them analogues of theorems for smooth surfaces in $\mathbb{P}^4$. This does not mean that proofs are just a mere imitation of those for $\mathbb{P}^4$. For example, Theorem 5.1, which is analogous to the Severi theorem in $\mathbb{P}^4$, is proved by looking at the geometry of lines in $\mathbb{P}^3$, so that not even a single idea of the proof of Severi is used. We also give in §4 a classification and description of the Hilbert scheme of smooth congruences of degree at most nine. At the end of §2 and §4, we add an appendix with partial results and conjectures, in order to show the numerous possibilities for future research with congruences.

The distribution of this work is as follows:

-In §0 we just give some preliminary definitions, as well as some general result that we will use frequently.

-We devote §1 to give general properties of vector bundles on G and we introduce the most important ones, that will be used later, especially in §4 to give resolutions of the ideal sheaves of congruences. We also give a new and shorter proof of the fact that the only indecomposable vector bundles on G not having intermediate cohomology are the line bundles and the twists of the universal bundles. This result, proved in algebraic terms by Knörrer (see [29]) with great generality, was independently proved for G by the second author by purely geometric means, but not published until he obtained a proof for all smooth quadrics (see [2]).

-In §2, after proving some known general results on congruences and several technical results that we will use to study the Hilbert scheme of congruences, we add an appendix where we prove some partial results related to a conjecture of Dolgachev on the semistability of the restriction to smooth congruences of the universal vector bundles.

-Section §3 consists of the development of a new theory of linkage for congruences. It is not exactly analogous to the known theory for projective varieties, since in our theory the role of complete intersections is played not only by them, but also by what we call spinor congruences (that are zero locus of sections of twists of the universal quotient bundle $E_2^\vee$). We show that our definition is the right one by proving analogous results to those for linkage in projective spaces. We prove, for example, that even liaison classes are in 1-1 correspondence with classes of vector bundles on G not having first cohomology spaces after tensoring with any line bundle and twists of E_2 (where two bundles are in the same class when they differ by a twist after removing their components of the form $\mathcal{O}_G(l)$ or $E_2(m)$). As an easy corollary of this theory, we get resolutions for the ideal sheaves of smooth congruences lying in a linear complex.

-In §4 we give a classification of smooth congruences of degree at most nine, giving a resolution of their ideal sheaves in G (thus providing a complete information on their cohomology, in particular we know the postulation of the congruences) and using it to describe their Hilbert scheme.

Section §4.1 corresponds to congruences of degree less than or equal to eight, whose classification is essentially due to Fano (see [11]) under different conditions. Also Papantonopoulou gives in [34] (with a slight mistake) a list of possible smooth congruences of degree at most eight (some members of her list actually do not exist). Our new contribution, besides giving the precise list of smooth congruences, is the description of their Hilbert scheme. These results appeared in [1]. New information on the restriction of the universal vector bundles to some of these congruences also appears in this section.

In section §4.2, devoted to smooth congruences of degree nine, something similar happens, since their classification was obtained by Verra (with a numerical mistake in the computation of the invariants of the one lying in a linear complex). We add as an appendix two more sections. In section §4.3 we selected several examples of more congruences (some of them in a quite incomplete way) and in section §4.4 we state some known results on congruences that we will need later or just that are interesting to get a global view of congruences).

-Section §5 contains a proof of what is the analogous for congruences of the Severi theorem in $\mathbb{P}^4$. More precisely, we prove that, excepting an explicit list of five types of congruences, no other smooth congruence can be obtained as a projection of a surface in $Gr(1,4)$, the Grassmann variety of lines in $\mathbb{P}^4$. We complete this section using this result to classify those smooth congruences such that the restriction to them of the universal quotient bundle decomposes as a direct sum of two line bundles.

-Finally, section §6 solves a conjecture of Robin Hartshorne made for both $\mathbb{P}^4$ and G, stating that only a finite number of families of smooth surfaces in these spaces correspond to rational surfaces. This problem was solved by Ellingsrud and Peskine for $\mathbb{P}^4$ (see [10]), proving a stronger result which has as an easy corollary that all but a finite number of families of smooth surfaces in $\mathbb{P}^4$ are of general type. Their result can be stated in several different ways, as Christian Peskine pointed out to us. In particular, we prove that there exists only a finite number of families of smooth congruences S having a fixed Euler-Poincaré characteristic $\chi(\mathcal{O}_S)$.

Our proof is essentially a translation of theirs to G. The new and original part is mostly in section §6.1, where we obtain bounds for the genus of curves in Q_3 (hence for the sectional genus of congruences) depending on their postulation. Such a bound for curves in $\mathbb{P}^3$ was obtained by Gruson and Peskine by restricting to general plane sections and defining for it a series of ordered numbers which are each proved to differ by at most one with its neighbors. The solution for Q_3 is not so easy, and the trick that one has to use is to restrict not to smooth quadrics, but to quadratic cones, and define a series of ordered numbers that each differ by at most two with its neighbors. We also want to mention that the calculations in section §6.2 were made in collaboration with Manuel Pedreira.

We want to thank Christian Peskine for his continuous encouragement and help in the preparation of this work. We also shared useful conversations with G. Ellingsrud, K. Ranestad, A. Aure and A. Verra. We thank also M. Gross, who had us always informed on his progress in the topic, and also has helped a lot in correcting the grammar of the paper. Both authors have been supported by CAICYT grant PB86-0036 during the preparation of this work.

§0. PRELIMINARIES

We will denote with G the Grassmann variety $Gr(1,3)$ of lines in the projective space $\mathbb{P}^3=\mathbb{P}(V)$ over $\mathbb{C}$. Via the Plücker embedding, we can also view G as a smooth quadric in $\mathbb{P}^5= \mathbb{P}(\Lambda^2 V)$. The Chow ring of G is very well-known, and we have

-$A^1(G) = \mathbb{Z}\eta_1$ where η_1 is the class of the hyperplane section of G in $\mathbb{P}^5$. If the hyperplane is tangent at a point l of G, then its intersection with G is a cone with vertex l that corresponds to the Schubert cycle of lines of $\mathbb{P}^3$ that meet the line L represented by the point l. This is called a *special linear complex.* A hypersurface of G having class $d\eta_1$ in $A^1(G)$ is called a *complex of degree d.*

-$A^2(G) = \mathbb{Z}\eta_{2,1}\oplus \mathbb{Z}\eta_{2,2}$ where $\eta_{2,1}$ is the Schubert cycle of lines of $\mathbb{P}^3$ passing through a fixed point (also called an *α-plane,* since it is a plane inside $\mathbb{P}^5$) and $\eta_{2,2}$ is the Schubert cycle of lines of $\mathbb{P}^3$ contained in a fixed plane (also called a *β-plane*). Each plane of G is either an α-plane or a β-plane.

-$A^3(G) = \mathbb{Z}\eta_3$ where η_3 is the Schubert cycle of lines of $\mathbb{P}^3$ contained in a fixed plane and passing through a fixed point of the plane. This represents a line in $\mathbb{P}^5$ and, conversely, all lines of G admit such a geometric interpretation in $\mathbb{P}^3$.

-$A^4(G) = \mathbb{Z}\eta_4$ where η_4 is, of course, the class of a point of G.

The multiplicative structure is given by

$$\eta_1^2= \eta_{2,1}+ \eta_{2,2}$$
$$\eta_1\eta_{2,1}= \eta_3$$
$$\eta_1\eta_{2,2}= \eta_3$$
$$\eta_1\eta_3= \eta_4$$
$$\eta_{2,1}^2= \eta_4$$
$$\eta_{2,2}^2= \eta_4$$

A cycle in $A^{i}(G)$ can, therefore, be denoted by an integer number, except for the case i=2, where we will use pairs of integers (a_1,a_2) to denote the class $a_1\eta_{2,1}+a_2\eta_{2,2}$. An element in the Chow ring of G will be written in polynomial form as

$$a_0+a_1t+(a_{2,1},a_{2,2})t^2+a_3t^3+a_4t^4$$

With this convention, since G is a quadric in $\mathbb{P}^5$ we can write its canonical line bundle as $\omega_G=\mathcal{O}_G(-4)$.

Notations. Throughout this work, we will use the following conventions:

An element of G will be denoted with a small latin letter (e.g. l) and the line in $\mathbb{P}^3$ it represents will be denoted by the corresponding capital letter (L in our example).

For any subvariety X of G, $\mathcal{J}_X$ will denote the ideal sheaf of X in G.

If S be a smooth surface in G, we use the following notations for its invariants:

○The *order* d_1 of S is defined to be the number of lines of the congruence passing through a fixed general point of $\mathbb{P}^3$.

○The *class* d_2 will be the number of lines of the congruence contained in a fixed general plane.

○We will denote with d the *total degree* $d=d_1+d_2=H^2$ of S, that is its degree as a surface in $\mathbb{P}^5$ (H denotes the hyperplane class of S, i.e. the class of its intersection with a general linear complex).

○The Euler-Poincaré characteristic of $\mathcal{O}_S$ will be denoted by $\chi=1-q+p_g=1+p_a$, where $q=h^1(\mathcal{O}_S)$ is the *irregularity* of S, $p_g=h^2(\mathcal{O}_S)$ is the *geometric genus* and p_a is the *arithmetic genus.*

○We use the symbol K to represent the canonical class of S

○We denote by π the *sectional genus* of S, i.e., the genus of the intersection of S with a general linear complex. By the adjunction formula, $2\pi-2=H^2+KH$.

There is an isomorphism $G(1,\mathbb{P}^3)\xrightarrow{\cong}Gr(1,\mathbb{P}^{3\vee})$ mapping each line in $\mathbb{P}^3$ intothe pencil of planes containing the line. Hence, any congruence in $G(1,\mathbb{P}^3)$ of bidegree (d_1,d_2)

produces a congruence in $G(1,\mathbb{P}^{3\vee})$ of bidegree (d_2,d_1) with the same invariants. We will refer to this fact as *duality*.

Finally, we state here some general results that we will use several times throughout this work (in §4.4 we give a similar list of results concerning congruences, once the necessary ingredients are introduced).

[46] Proposition 1.1 (2). *Let X, Y, Z, S be smooth schemes appearing in a diagram*

$$\begin{array}{ccc} W & \longrightarrow & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\ f\ } & Z \\ {\scriptstyle \pi}\downarrow & & \\ S & & \end{array}$$

where $W = X\times_Z Y$ and f is a smooth map. Then, there is a dense open set U_1 in S such that for each s in U_1 the fiber W_s of $\pi\circ h$ (that is $X_s\times_Z Y$) is either empty or smooth. (The ground field here must have characteristic zero).

[23] Proposition 9.5. *A vector bundle F on $\mathbb{P}^r$ is a direct sum of line bundles if and only if $H^i(F(l))=0$ for all integers l and $0<i<r$.*

[31] Theorem 2. *Let V a complete normal variety of dimension at least two (over an algebraically closed field of characteristic zero) Let $\mathcal{L}$ be an invertible sheaf on V such that, for large n, $\mathcal{L}^n$ is spanned by its sections. Let these sections define the morphism $V \longrightarrow W$. Then, $H^1(\mathcal{L}^{-m}) = 0$ for all $m\geq 1$ if and only if $\dim(W)>1$.*

The easy corollary we will apply is the following: *Let $\tilde{X}$ be the normalization of a projective surface X (in our case X will be a surface in a smooth quadric of $\mathbb{P}^4$). If we denote with $\mathcal{O}_{\tilde{X}}(1)$ to the pull-back to $\tilde{X}$ of $\mathcal{O}_X(1)$, then $H^1(\mathcal{O}_{\tilde{X}}(-m)) = 0$ for all $m\geq 1$.* (Just apply the above result making $\mathcal{L} = \mathcal{O}_{\tilde{X}}(1)$ and $n=1$).

We will also use a slight generalization of Mumford- Castelnuovo criterion (see Prop. 1.1).