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NONINVARIANT BASE CHANGE IDENTITIES

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Abstract. – We prove, in the cyclic base change situation for the group $GL(n)$, an identity between noninvariant trace formulas for pairs of strongly associated functions. We construct sufficiently many such pairs of functions in order to get a new proof of the existence of base change for automorphic representations of $GL(n)$ over a number field. Our proof is more direct and elementary than Arthur and Clozel's one, although based on a similar method: a trace formula identity.

Résumé. – On établit, dans le cas du changement de base cyclique pour le groupe $GL(n)$, une identité entre les formules des traces non-invariantes pour les paires de fonctions fortement associées. Nous construisons assez de telles paires de fonctions pour en déduire une preuve nouvelle de l'existence du changement de base cyclique pour les représentations automorphes de $GL(n)$ sur un corps de nombre. Notre preuve est plus directe et plus élémentaire que celle d'Arthur et Clozel quoique basée sur une méthode analogue: une identité de formule de traces.

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INTRODUCTION

Let G be an inner form of a reductive quasi-split group H , defined over a global field F . Let E/F be a finite field extension. According to the Langlands philosophy there should exist a base change correspondence between automorphic representations of $H(\mathbb{A}_F)$ and $G(\mathbb{A}_E)$. To prove the existence of such a correspondence when E/F is a cyclic extension of degree ℓ , one may use a technique due to Saito Shintani and Langlands : a term by term comparison of two trace formulas.

In the case of number fields, this has been worked out for inner forms of $GL(n)$ in [AC] and for unitary group in three variables attached to a quadratic extension E/F in [Rog]. Let θ be a generator for the Galois group of E over F . Roughly speaking, one first shows the equality of the geometric expansions of the stable trace formula for H and of the stable trace formula for $L = \mathrm{Res}_{E/F}\, G \rtimes \theta$ when applied to pairs (f, ϕ) of *associated functions* $f \in C_c^\infty(H(\mathbb{A}_F))$ and $\phi \in C_c^\infty(L(\mathbb{A}_F))$. The correspondence $\phi \mapsto f$ is a particular case of twisted endoscopic transfer whose existence has to be established; moreover one has to show that *association* is compatible with base change for functions in the unramified Hecke algebras. This is the *fundamental lemma* for the stable base change, now proved in general in [Clo] for fields of zero characteristic, and in [Lab2]. This allows to separate unramified infinitesimal characters (i.e. characters of the unramified Hecke algebras) and one deduces from this the matching of the various terms in the spectral expansions of the two trace formulas; this yields the base change correspondence for automorphic representations.

Even in the case $H = GL(n)$ which is particularly simple since, for such a group, conjugacy and stable conjugacy coincide, the term by term comparison of the geometric expansions in the two trace formulas applied to pairs of associated functions (f, ϕ) is not straightforward. The main difficulty arises from the following fact : the trace formula is obtained by a truncation process which is noninvariant under conjugacy, while the concept of association allows only comparison between invariant distributions. The standard procedure is to put the trace formula into an invariant form. The existence of such an invariant form is proved in [A8] but uses long and difficult prerequisites ([A6], [A9], etc.) Moreover, it is not easy to compare the invariant distributions $I_M(\gamma, f)$ and $I_{M^L}(\delta, \phi)$ – constructed from the weighted orbital integrals $J_M(\gamma, f)$ and $J_{M^L}(\delta, \phi)$ – that show up in the invariant trace formula, since they are defined in a rather implicit way if $M \neq H$. Another difficulty is that the contributions (invariant or not) of non-semisimple conjugacy classes are very complicated and a direct comparison seems hopeless. These are the reasons for the quite intricated and difficult arguments in [AC] chapter 2.

Our aim is to suggest a way to bypass these difficulties and to test this program in the case of $GL(n)$. The main simplification is that we compare directly the primitive – noninvariant – form of the two trace formulas. This is made possible by using a noninvariant endoscopic transfer we call *strong association.*

Another simplification is that we do not use any analysis, locally or globally, of the behaviour of orbital integrals near the singular set. Globally this is because we may use, at some place, pairs of functions with regular support: doing so we kill the singular terms in the geometric expansion of the trace formula, but fortunately we do not lose any spectral information. Locally, besides the noninvariant fundamental lemma for units in the unramified Hecke algebras, we only need the noninvariant endoscopic transfer for functions with regular support; this is enough thanks to the very strong finiteness results which follow from the rigidity of cuspidal automorphic representations of $G = GL(n)$. Unfortunately, for other groups, such finiteness results may not be available right away and it might turnout that one would have to rely more on noninvariant harmonic analysis for groups over local fields.

This paper is an expanded version of a preprint [Lab3] that has been circulated in 1992. We have strived to make the paper self-contained from our starting point: the trace formula as obtained in the early papers by Arthur. To make the paper more accessible we even review the definition of the distributions that show up in the trace formula and we sketch the proof of the properties we need. As a result, most of the material in chapter I and a large part of chapter II is borrowed from Arthur's papers, but we believe it more convenient for the reader to have it reviewed in some detail here. Many techniques are borrowed from [AC], this is acknowledged case by case, but we have tried not to rely on references to [AC]. This is so with few exceptions, where we have only quoted some results whose proof do not depend of the main body of [AC]: in I.8.2, the first step in the construction of a function on a Cartan subalgebra is borrowed from the chapter 2 of [AC] but this is an elementary result; in III.1.5 we refer to the first few pages of the first chapter of [AC] for the classical properties of the norm map; the most significant borrowed result is the compatibility of local L-functions with the local base change, the proof of which occupies a large part of the last two sections of the first chapter of [AC]; this is our proposition VI.5.2. Let us now describe the contents of the paper.

In chapter I we give the definitions and review the basic properties of the distributions that show up in the geometric and the spectral expansions of the noninvariant trace formula. The last two sections contain new material.

In chapter II we review the noninvariant trace formula itself. The absolute convergence of the spectral expansion of the trace formula is stated as a conjecture (Conjecture A) in section II.2. We hope that conjecture A will follow from work in progress by W. Müller. We recall an estimate, due to Arthur, that can be used to separate infinitesimal characters, via multipliers, at archimedean places. This estimate is a weak form of the conjectural absolute convergence of the spectral expansion. In section II.4, a particular case of conjecture A which is enough for our needs is established.

In chapter III we begin the study of base change; to avoid stabilization problems we restrict ourselves to groups G that may show up as Levi subgroups of inner forms of $GL(n)$. We introduce a refined version of the concept of association: we consider pairs of functions f and ϕ such that not only orbital integrals but also weighted orbital integrals $J_M(\gamma, f)$ and $J_{M^L}(\delta, \phi)$ are equal, if γ is the norm of δ , at least when these elements are regular semisimple. Moreover the weighted orbital integrals of f have to satisfy some vanishing properties if γ is not a norm. Such pairs of functions will be called *strongly associated.* The best we hope, as regards this noninvariant endoscopic transfer, is stated as conjecture B. The existence of pairs of strongly associated functions with regular support is easy to establish. At the end of chapter III we prove the conjecture B for split places.

In chapter IV we study unramified places: we have to show that the noninvariant endoscopic transfer is compatible with the base change map between unramified Hecke algebras. The key observation is that, thanks to a result of Kottwitz, a noninvariant fundamental lemma holds for units in the unramified Hecke algebras and yield pairs of strongly associated functions. We first recall the definition of elementary functions and we show that they are closely related to functions bi-invariant under an Iwahori subgroup. We show that elementary functions give rise to pairs of strongly associated functions. Moreover, strong association of elementary functions is compatible with base change for weighted characters; this allows to prove a noninvariant form of the fundamental lemma for all functions in the unramified Hecke algebra. Most of the proof of these last two results is postponed to chapter V.

In chapter V we state our base change identity. The matching of the regular semisimple terms in the two trace formulas for pairs of strongly associated functions is obvious. For pairs of strongly associated functions (f,ϕ) , with regular support at one place, the contributions of non-semisimple conjugacy classes vanish and we get the equality of two noninvariant trace formulas :

$$J^H(f) = J^L(\phi) \ .$$

As a first consequence of this identity we prove a twisted version of a noninvariant form of Kazdan's density theorem. Then we show how to use conjecture B2 to refine the spectral identity for pairs of strongly associated functions by separating infinitesimal characters at archimedean places. This is applied to the proof of the noninvariant fundamental lemma. The proof is based on a refinement of the local-global argument used in [Lab2].

In chapter VI, we deal with the base change of automorphic representations. We first refine the spectral identity for pairs of strongly associated functions by separating infinitesimal characters at unramified places. If conjecture B2 holds (in particular if $G = GL(n)$ and E/F splits over archimedean places) we may first separate the archimedean infinitesimal characters and we are left, for a given conductor, with a finite set of automorphic representations; using pairs of associated elementary functions or the noninvariant fundamental lemma, we may separate finite sum of unramified infinitesimal characters. In general, since we do not know that strong association at archimedean places is compatible with multipliers, we have to separate infinite families of unramified infinitesimal characters. This could be done directly, using the fundamental lemma, if we knew that the spectral expansion of trace formula is absolutely convergent (conjecture A); the particular case established in chapter II is enough to conclude if we may choose the normalizing factors for intertwining operators to be compatible with the weak base change. To finish the proof of the existence of base change and of his properties for $GL(n)$ we use in an essential way, as in [AC], the strong finiteness properties that follow from Jacquet-Shalika's theorem on L-functions of pairs, in particular the rigidity (or strong multiplicity one) for cuspidal automorphic representations of $GL(n)$. Thus we obtain a new proof of Arthur-Clozel's theorem. Our result is slightly more general since, thanks to Mœglin-Waldspurger's description of the discrete spectrum, it is no more necesary to restrict oneself to automorphic representations "induced from cuspidal". For inner forms we cannot use a priori the rigidity, although it can be deduced from the properties of the endoscopic correspondence. Hence, to extract the expected informations on the endoscopic correspondence from our noninvariant trace formula identity, without using the rigidity, one would need either a weaker form of it, namely some a priori finiteness result (conjecture C), or further local results.

We observe that, if the trace formula for groups over function fields were available, our proof should extend easily to the case where ℓ is prime to the characteristic of the function field.

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