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# On the search of genuine $p$-adic modular $L$-functions for $GL(n)$. With a correction to : on $p$-adic $L$-functions of $GL(2) \times GL(2)$ over totally real fields

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# On the Search of Genuine $p$-adic Modular $L$-Functions for $GL(n)$

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**Abstract** — The purpose of this monograph is to state several conjectures concerning the existence and the meromorphy of many variable $p$-adic $L$-functions attached to many variable Galois representations (for example having values in $GL_n(\mathbb{Z}_p[[X_1, \ldots, X_r]])$) and to present some supporting examples for the conjectures. Our discussion in the earlier sections is therefore quite speculative, but towards the end, we gradually make things more concrete.

**Résumé** — Le but de cette monographie est de formuler quelques conjectures concernant l'existence et la méromorphie des fonctions $L$ $p$-adiques de plusieurs variables attachées à des représentations galoisiennes de plusieurs variables (par exemple, à valeurs dans $GL_n(\mathbb{Z}_p[[X_1, \ldots, X_r]])$) et de présenter quelques exemples motivant nos conjectures. Nous commençons par une discussion assez spéculative, mais vers la fin, nous donnons des résultat plus concrets.

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# Contents

# 1. Introduction

The purpose of this monograph is to state several conjectures concerning the existence, a precise interpolation property and the meromorphy of many variable $p$-adic $L$-functions attached to many variable (irreducible) Galois representations (for example having values in $GL_n(\mathbb{Z}_p[[X_1, \dots, X_r]])$) and to present some supporting examples for the conjectures. Often in the non-abelian case, $p$-adic $L$-function is determined up to unit multiples. We would like to identify a precise interpolation property necessary to determine the $p$-adic $L$ uniquely. This is important to have a non-abelian generalization of classical class number formulas and limit formulas which gives a direct connection between the analytic $p$-adic $L$ and arithmetic objects. The theory of abelian $p$-adic $L$-functions is constructed out of a desire to better understand the class number formulas. Contrary to this, in the non-abelian case, it is ironic for us to be left in search of a $p$-adic $L$-function genuinely characterized by the Galois representation by which the generalized class number formula should be written down. Our discussion in the earlier sections is therefore quite speculative, but towards the end, we gradually make things more concrete.

Let us describe our idea. Let $p$ be a prime, and fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $F$ be a finite extension of $\mathbb{Q}$. We write $I$ for the set of all embeddings of $F$ into $\overline{\mathbb{Q}}$. We will later fix an embedding $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and take a field $K$ which is a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ containing the image under $i_p\sigma$ for every $\sigma \in I$. Write $\mathcal{O}$ for the $p$-adic integer ring of $K$. We extend $\sigma$ to an isomorphism: $\overline{F} \cong \overline{\mathbb{Q}}$ which we denote again by $\sigma$. For each $p$-adic place $\mathfrak{P}$ of $F$, we write $\mathcal{G}_{F_{\mathfrak{P}}}$ for the Galois group $\mathrm{Gal}(\overline{F}_{\mathfrak{P}}/F_{\mathfrak{P}})$, and if $\mathfrak{P}$ is induced from $i_p\sigma$, we identify it with the decomposition group at $\mathfrak{P}$ in $\mathcal{G}_F = \mathrm{Gal}(\overline{F}/F)$ via $i_p\sigma$. Let $T_n$ be the standard diagonal torus of $\mathrm{Res}_{\mathfrak{r}/\mathbb{Z}}\, GL(n)$ for the integer ring $\mathfrak{r}$ of $F$ which is split over $\mathcal{O}$. Then we consider a normal integral domain $\mathbb{I}$ finite (but not necessarily flat) over the completed group algebra $\mathcal{O}[[T_n(\mathbb{Z}_p)]]$. We assume that $\mathcal{O}$ is integrally closed in $\mathbb{I}$. Each $\mathcal{O}$-algebra homomorphism $P : \mathbb{I} \to K$ restricted to $\mathcal{O}[[T_n(\mathbb{Z}_p)]]$ induces a continuous character $\kappa(P)$ of $T_n(\mathbb{Z}_p) \to K^\times$. We call $P \in \mathit{Spec}(\mathbb{I})(K)$ *arithmetic* if it induces a weight (that is, an algebraic character) of the torus $T_n$

on an open neighborhood of the identity in $T_n(\mathbb{Z}_p)$. We want to study a continuous Galois representation $\varphi : \mathcal{G}_F \to GL_n(\mathbb{I})$ acting on $V(\cong \mathbb{I}^n)$ satisfying the following condition:

(A1) *There are arithmetic points $P$ densely populated in $Spec(\mathbb{I})(K)$ such that the Galois representation $\varphi_P = P \circ \varphi$ is the $p$-adic étale realization of a rank $n$ pure motive $M_P$ defined over $F$ with coefficients in a number field $E_P$ in $\overline{\mathbb{Q}}$.*

The number field $E_P$ depends on $P$, and the composite $E_\infty$ of all $E_P$ for $P$ satisfying (A1) is usually an infinite extension of $\mathbb{Q}$. We call points $P$ satisfying (A1) *motivic.* Densely populated motivic points determine an isomorphism $i_p : E_\infty \hookrightarrow \overline{\mathbb{Q}}_p$ such that $Tr(\varphi_P(Frob_{\mathfrak{l}}))$ for primes $\mathfrak{l}$ unramified for $\varphi_P$ generates a subfield of $i_p(E_P)$. We extend $i_p$ to $\overline{\mathbb{Q}}$. To get reasonable $p$-adic $L$-functions, we need to assume further

(A2$_\pm$) *For a dense subset of motivic points $P$, we have*

- (i) *The Tate twist $M_P(1)$ is critical in the sense of Deligne [15], and for each $p$-adic place $\mathfrak{P}$ of $F$, the restriction of $\varphi_P$ to $\mathcal{G}_{F_{\mathfrak{P}}}$ is of Hodge-Tate type;*
- (ii) *Writing $\mathcal{F}_P^\mp$ for the middle terms of the Hodge filtration of $H_{DR}(M_P)$ as in [15], for each $p$-adic place $\mathfrak{P} = i_p\sigma$ of $F$, there exists an $\mathbb{I}$-direct summand $V_{\mathfrak{P}}^\pm \subset V$ (independent of $P$) stable under $\mathcal{G}_{F_{\mathfrak{P}}}$ such that the comparison isomorphism of Faltings [21] induces: $V_{\mathfrak{P}}^\mp \otimes_{\mathbb{I},P} B_{HT} \cong \mathcal{F}_P^\mp \otimes_{F\otimes E, i_p\sigma\otimes i_p} B_{HT}$,*

where $B_{HT}$ is one of the Fontaine's rings [22] given as follows: Writing $\Omega$ for the $p$-adic completion of $\overline{\mathbb{Q}}_p$, we have $B_{HT} \cong \Omega[t, t^{-1}]$ on which $\mathcal{G}_{\mathbb{Q}_p}$ acts via the natural action on $\Omega$ and via the cyclotomic character on the indeterminate $t$. Since $V(\varphi_P)$ is of Hodge-Tate type, as a $\mathcal{G}_{F_{\mathfrak{P}}}$-module, writing $E_{P,\mathfrak{p}}$ for the topological closure of $i_p(E_P)$ in $\Omega$, $V(\varphi_P) \otimes_{E_{P,\mathfrak{p}}} \Omega \cong \bigoplus_{1\le i\le n} \Omega(m_i)$ for integers $m_i$ depending on $i_p\sigma$, where $\Omega(m) = \Omega t^m$ as a graded component of $B_{HT}$. We call $(m_1, \ldots, m_n)$ the Hodge-Tate twist of $V(\varphi_P)$ at $\sigma$. We will show that the condition (ii) is equivalent to the admissibility condition of Panchishkin ([56] Section 5) for $M_P$ if $M_P$ is crystalline at $p$. A similar condition is stated and studied in [28] p.217 in terms of Selmer groups. In particular, the existence of $V_{\mathfrak{P}}^+$ in (A2$_-$) plays an important role in defining the Selmer group. We require by (ii) an analytic coherence of $\mathcal{F}_P^\pm$ with respect to analytically varying $P$. The density assumption included in (A2$_\pm$) about motivic critical points is important to guarantee the uniqueness of the $p$-adic $L$-function. We also assume

(unr) *$\varphi$ is unramified at almost all places of $F$.*

The (conjectural) $\mathbb{I}$-adic representations arising from cohomological modular forms on $GL(n)_{/F}$ further satisfies an additional condition besides (A1-2$_\pm$). Since $T_n \cong \mathbf{G}_m^n$ over $F$, $X(T_n)$ is isomorphic to the product of $n$ copies of the free module $\mathbb{Z}[I]$

generated by the set $I$ of all embeddings of $F$ into $\overline{\mathbb{Q}}$. Thus we can associate to each arithmetic point $P$, an $n$-tuple $(m_1(P), \dots, m_n(P))$ of elements in $\mathbb{Z}[I]$. Then the condition is

(A3) *The Hodge-Tate twist of $\varphi_P$ restricted to $\mathcal{G}_{F_{\mathfrak{P}}}$ for the p-adic place $\mathfrak{P}$ induced by $i_p\sigma$ is given by $(m_1(P)_{c\sigma}, m_2(P)_{c\sigma}, \dots, m_n(P)_{c\sigma})$ for each $\sigma \in I$,*

where for $m \in \mathbb{Z}[I]$, we have written $m = \sum_\sigma m_\sigma \sigma$, and $c$ stands for the complex conjugation which we specify below choosing an embedding $i_\infty$ of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. This last condition (A3) might be always satisfied by representations satisfying (A1-$2_\pm$) after modifying the algebra structure over $\mathcal{O}[[T_n(\mathbb{Z}_p)]]$ and is equivalent to

(A3′) *The Hodge types of $H_B({}^{i_\infty\sigma}M_P)\otimes_{E,i_\infty}\mathbb{C}$ is given by $\{(m_i(P)_\sigma, m_{n+1-i}(P)_{c\sigma})\}_i$,*

where $H_B({}^{i_\infty\sigma}M_P)$ is the Betti realization of $M_P$ at $i_\infty\sigma$. Anyway, we call the representation satisfying (A1-3) and (unr) an arithmetic Galois representation.

As conjectured by Deligne [15], for each motivic point $P$ as in (A$2_-$), we expect to have a well defined special value

$$\frac{L(1, M_P)}{\mathbf{c}^+(M_P(1))} \in E_P \subset E_P \otimes_{\mathbb{Q}} \mathbb{C}$$

for Deligne's period $\mathbf{c}^+(M_P(1))$ in $(E \otimes_{\mathbb{Q}} \mathbb{C})^\times$. We now choose an embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and write $\frac{L_{i_\infty}(1,M_P)}{c^+_{\infty,i_\infty}(M_P(1))}$ for the $i_\infty$-component of $\frac{L(1,M_P)}{\mathbf{c}^+(M_P(1))}$. An element $L$ of the quotient field $\mathbb{K}$ of $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega$ can be regarded as a ($p$-adic) meromorphic function on $Spec(\mathbb{I})(\Omega)$ assigning the value $L(P) = P(L)$ to $P \in Spec(\mathbb{I})$, where $\mathcal{O}_\Omega$ is the $p$-adic integer ring of $\Omega$. We call such functions "meromorphic", because it is a ratio of elements in $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega$ and hence a ratio of $p$-adic analytic functions on a non-empty Zariski open subset of $Spec(\mathbb{I})$ (or more precisely, on the formal completion of a non-empty open subscheme of $Spec(\mathbb{I})$ along its fibre over $p$). Since $\mathbb{I}$ is a normal integral domain, if a meromorphic function in the above sense is everywhere defined, it is actually an element in $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega$ (and hence is an Iwasawa function when $\mathbb{I}$ is an irreducible component of $\mathcal{O}[[T(\mathbb{Z}_p)]]$ for a quotient torus $T$ of $T_n$). We expect that this is the case when $\varphi$ modulo the maximal ideal of $\mathbb{I}$ is absolutely irreducible. We fix a choice of $S = \{V^+_{\mathfrak{P}}\}_{\mathfrak{P}}$. If there exists an element $L_{S,i_\infty}(\varphi) \in \mathbb{K}$ satisfying the following condition for all motivic points $P$ as in (A1-$2_-$):

$$\text{(Int)} \qquad i_p^{-1}\left(\frac{L_{S,i_\infty}(P,\varphi)}{c^+_{p,i_\infty}(M_P(1))}\right) = *\, i_\infty^{-1}\left(\frac{L_{i_\infty}(1, M_P)}{c^+_{\infty,i_\infty}(M_P(1))}\right)$$

for a constant "$*$", we call $L_{S,i_\infty}(\varphi)$ a *genuine* $p$-adic $L$-function of $\varphi$ of type $S$. The exact form of the constant "$*$" is known by Coates, Perrin-Riou and Panchishkin, when $M_P$ is crystalline at $p$ and $\varphi$ contains all cyclotomic deformation of $M_P$ (see, [60], [56] Section 6, and Conjecture 4.2.1 in Chapter 4 in the text). Here $c^+_{p,i_\infty}(M_P(1))$