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THE b-PSEUDODIFFERENTIAL CALCULUS ON GALOIS COVERINGS AND A HIGHER ATIYAH-PATODI-SINGER INDEX THEOREM

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Abstract. Let $\Gamma \to \widetilde{M} \to M$ be a Galois covering with boundary. In this paper we develop a b-pseudodifferential calculus on the noncompact manifold $\widetilde{M}$. Our main application is the proof of a higher Atiyah-Patodi-Singer index formula, for a generalized Dirac operator $\widetilde{D}$ on $\widetilde{M}$, under the assumption that the group Γ is of polynomial growth with respect to a word metric and that the L^2-spectrum of the boundary operator $\widetilde{D}_0$ has a gap at zero. Our results extend work of Atiyah-Patodi-Singer, Connes-Moscovici and Lott.

Résumé. Soit $\Gamma \to \widetilde{M} \to M$ un revêtement Galoisien à bord. Dans cet article nous développons un b-calcul pseudodifférentiel sur $\widetilde{M}$. Ceci nous permet de prouver un théorème de l'indice supérieur d'Atiyah-Patodi-Singer, pour un opérateur de Dirac $\widetilde{D}$ sur $\widetilde{M}$, sous l'hypothèse que le groupe Γ est à croissance polynomiale par rapport à une métrique des mots et que zéro est un point isolé du spectre L^2 de l'opérateur de bord $\widetilde{D}_0$. Notre résultat généralise des travaux d' Atiyah-Patodi-Singer, Connes-Moscovici et Lott.

Key words: Dirac operators, Galois coverings, C^*-algebras, b-calculus, noncommutative de Rham homology, finite propagation speed estimates.

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0. Introduction.

One of the fundamental tools in the development of index theory for elliptic operators has been the use of heat-kernel techniques. As this introduction in meant for a wide audience, we briefly recall the main point of this approach. Suppose, for simplicity, that M is an even dimensional closed *spin* compact manifold with a fixed spin structure. Let $S = S^+ \oplus S^-$ be the bundle of spinors and let D be the Dirac operator associated to the given spin structure. The operator D is formally self-adjoint and odd with respect to the $\mathbb{Z}_2$-grading; thus $D^{\pm} : C^\infty(M, S^{\pm}) \to C^\infty(M, S^{\mp})$ and $D^- = (D^+)^*$. The heat operator of the Dirac laplacian, $\exp(-t\, D^2)$, is a smoothing operator for each $t > 0$. Thus the Schwartz kernel of $\exp(-t\, D^2)$, the *heat kernel*, is smooth on $M \times M$ and it is therefore trace class acting on the Hilbert space of L^2 sections of S. Consider the supertrace of $\exp(-tD^2)$, $\mathrm{STr}\,(\exp(-tD^2)) \equiv \mathrm{Tr}(\exp(-t\,D^-D^+)) - \mathrm{Tr}(\exp(-t\,D^+D^-))$. The vanishing of the trace on commutators implies that this difference does not depend on t, thus

$$\frac{d}{dt}(\mathrm{STr}\,(\exp(-tD^2))) = 0 \qquad (0.1).$$

Moreover, by Lidski's theorem, it is given by the difference of the integrals of the two heat kernels over the diagonal Δ of $M \times M$. It is well known that as $t \to +\infty$ the heat operator converges exponentially to the orthogonal projection onto the null space of D^2. This implies that $\mathrm{STr}\,(\exp(-tD^2))$ converges exponentially to the supertrace of the projection onto the null space of D^2 which is easily seen to be the index of D^+. On the other hand as $t \to 0^+$ the heat kernel restricted to the diagonal converges itself to a density on $\Delta \equiv M$ which is *explicitly computable.* We denote this density by $\mathrm{AS}_{[n]}$, n being the dimension of the manifold M and AS being an explicit differential form constructed out of the riemannian curvature tensor . The index theorem for D^+ then follows by equating the integral over $\Delta \equiv M$ of this explicit geometric expression with the supertrace of the projection onto the null space of D^2 (which is the index of D^+). Here formula (0.1) has been used. Thus

$$\mathrm{ind}(D^+) = \int_M \mathrm{AS}\,.$$

What we have just explained is a sketch of the proof of the *local* index theorem for Dirac operators (see [ABP][G][BGV]).

The fascinating idea of using the heat equation to investigate the index of Dirac operators (due to McKean and Singer in its first formulation) opened the way to a variety of extensions of the original results of Atiyah and Singer, some of which will be now recalled.

In a fundamental series of articles, Atiyah, Patodi and Singer [APS 1,2,3] extended the results of [AS 1,3] to Dirac operators on manifolds with boundary.

Thus suppose now that M has a boundary ∂M and that the riemannian metric is of product type near the boundary. The Dirac operators $D^\pm$ can be written, near the boundary, as $\pm\,\partial/\partial u + D_0$ with u equal to the normal variable to the boundary and D_0 the Dirac operator on ∂M. The operator D_0 is elliptic and essentially self-adjoint. Let $\Pi_\geq$ be the spectral projection corresponding to the non-negative eigenvalues of D_0 and let

$$C^\infty(M, S^+, \Pi_\geq) = \{\, s \in C^\infty(M, S^+) \mid \Pi_\geq(s|_{\partial M}) = 0\,\}.$$

The Atiyah-Patodi-Singer theorem [APS 1] states that the operator D^+ acting on Sobolev completions of $C^\infty(M, S^+, \Pi_\geq)$ (we denote this operator by $D^+_{\Pi_\geq}$) is a Fredholm operator with index

$$\mathrm{ind}(D^+_{\Pi_\geq}) = \int_M \mathrm{AS} - \frac{1}{2}(\eta(D_0) + \dim \mathrm{null}\, D_0).$$

Here $\eta(D_0)$ is the eta invariant of the self-adjoint operator D_0. It is a spectral invariant that measures the *asymmetry* of the spectrum of D_0. It is defined as the value at $s = 0$ of the meromorphic continuation of the complex function

$$\sum_{\lambda \neq 0} \operatorname{sign} \lambda \, |\lambda|^{-s} \quad \Re s >> 0$$

with λ running over the eigenvalues of D_0. Equivalently, using the Mellin transform,

$$\eta(D_0) = \frac{2}{\sqrt{\pi}} \int_0^\infty \operatorname{Tr}(D_0 \, e^{-(t\, D_0)^2}) dt \equiv \int_0^\infty \eta(D_0)(t) dt. \tag{0.2}$$

The proof of the Atiyah-Patodi-Singer theorem relies heavily on the heat-kernel method.

The Atiyah-Patodi-Singer index theorem has seen a number of reformulations and alternative proofs. Among the latest contributions to the subject we mention here the *b*-calculus approach of Melrose [Me] (see also [P1][MeNi]). In this new approach microlocal techniques are used in order to give an elegant and conceptually simple proof of the original result of Atiyah-Patodi-Singer (in fact for metrics which are more general then those, product-like near the boundary, considered in [APS 1]). We refer the uninitiated reader to the introduction of [Me] for a very readable summary of the main ideas surrounding the *b*-calculus proof.

We come now to Bismut's fundamental proof of the *local* family index theorem for Dirac operators on closed manifolds [B]. Given a smooth family of Dirac operators $D = (D_z)_{z \in B}$ acting on $C^\infty(M_z, S_z)$ and parametrized by a compact manifold B, we can consider the associated (regularized) index bundle $\operatorname{Ind}(D) = [\operatorname{null}(D^+)] - [\operatorname{null}(D^-)]$, an element in the K-theory $K^0(B)$ of the base B, and the associated Chern character $\operatorname{Ch}(\operatorname{Ind}(D))$, a cohomology class in $H^{\text{even}}(B, \mathbb{R})$. From an algebraic point of view the index bundle can be seen as the formal difference of two finitely generated projective $C^0(B)$ modules, $C^0(B)$ denoting the algebra of continuous functions on B (see [A]). Thus $\operatorname{Ind}(D) \in K_0(C^0(B))$, with $K_0(C^0(B))$ equal to the 0th algebraic K-group of $C^0(B)$. This point of view will be exploited later

The problem is once again to give a geometric formula for $\operatorname{Ch}(\operatorname{Ind}(D))$, an *a priori* analytic object. The cornerstone of Bismut's treatment of the family index theorem is the use of the superconnection formalism (see also [Q]). Instead of considering the family of Dirac laplacians (D_z^2) one considers a family of generalized laplacians *with differential form coefficients.* This new family is manufactured out of a superconnection on the infinite dimensional bundle, over B, whose fiber at $z \in B$ is equal to $C^\infty(M_z, S_z)$. The fiber supertraces of the heat-kernels associated to this new family of generalized laplacians combine to give a *smooth* differential form on B. We denote by K_t the family of heat-kernels and by $\mathrm{STr}(K_t)$ this smooth differential form. Bismut then proves that $\mathrm{STr}(K_t)$ satisfies the following properties

(i) It is a closed differential form $\forall t > 0$.
(ii) It does not depend on t modulo exact forms: $d/dt(\mathrm{STr}(K_t)) = d_B\, \alpha_t$.
(iii) It is explicitly computable as $t \to 0^+$.
(iv) It represents $\mathrm{Ch}(\mathrm{Ind}(D))$ in $H^{\mathrm{even}}(B)$.

This last property can be proved directly as in Bismut's original argument or by showing, as in [BV][BGV], that the limit as $t \to +\infty$ of $\mathrm{STr}(K_t)$ converges as a differential form on B to the Chern character of the index bundle.

Using these properties the *local* version of the family index theorem follows. In particular

$$\mathrm{Ch}(\mathrm{Ind}(D)) = \int_{\mathrm{fibre}} \mathrm{AS} \quad \text{in } H^*(B).$$

Among the many implications of Bismut's heat-kernel treatment of the family index theorem we concentrate now on the family version of the Atiyah-Patodi-Singer index theorem. The first result in this direction is due to Bismut and Cheeger [BC 1,2,3]; (D_z) is now a family of Dirac operators on manifolds with boundary, parametrized by a compact smooth manifold B. In order for the family $D^+_{z,\Pi_\geq}$ to define a *smooth* (or even continuous) family of Fredholm operators it is necessary that the null spaces of the boundary operators $D_{0,z}$ are of constant dimension in $z \in B$. Notice that under this assumption they form a smooth vector bundle over B, $\mathrm{null}(D_0) \to B$. Moreover the index bundle $\mathrm{Ind}(D_{\Pi_\geq})$ is well defined and the following formula holds

$$\mathrm{Ch}(\mathrm{Ind}(D_{\Pi_\geq})) = \int_{\mathrm{fibre}} \mathrm{AS} - \frac{1}{2}(\hat{\eta} + \mathrm{Ch}(\mathrm{null}(D_0))) \quad \text{in } H^*(B) \qquad (0.3)$$

(the formula is fully proved in the invertible case in [BC 1,2] and stated in the constant rank case in [BC 3]; see [MP 1] for a complete proof of (0.3)). In this formula $\hat{\eta}$ is the eta form of Bismut-Cheeger; it is a higher version of the eta invariant, in the sense that the 0-degree component of $\hat{\eta}$ computed at $z \in B$ is equal to $\eta(D_{0,z})$. The Bismut-Cheeger eta form is defined in terms of the superconnection formalism by a formula similar to (0.2). The assumption that the operators of the family D_0 have null spaces of constant dimension plays a crucial role in the proof of the convergence of the integral.

The results of Bismut-Cheeger were improved in [MP 1,2]. The use of a new notion, that of *spectral section* associated to a self-adjoint family of elliptic operators (like D_0), together with the pseudodifferential b-calculus, allowed for the formulation and the proof of a *general* Atiyah-Patodi-Singer family index theorem, both in the even and in the odd dimensional case.

Suppose now, as in the beginning of this introduction, that M is a closed compact spin manifold. Let us denote by Γ the fundamental group $\pi_1(M)$ of M and by $\widetilde{M} \to M$ the universal covering of M. The Γ-manifold $\widetilde{M}$ is again spin with a Γ-invariant Dirac operator $\widetilde{D}$ acting on the section of a Γ-invariant spinor bundle $\widetilde{S}$. It is clear that $\widetilde{M}$ will be in general non-compact. There are two sets of objects that are determined by the appearance of the fundamental group of M.

First we can consider the classifying map $\nu : M \to B\Gamma$ associated to the Γ-bundle $\Gamma \to \widetilde{M} \to M$. For each cohomology class $[\beta] \in H^*(B\Gamma, \mathbb{C})$ we can then consider $\nu^*[\beta] \in H^*_{dR}(M)$ and the complex numbers

$$\int_M \mathrm{AS} \wedge \nu^*[\beta].$$

Recall also that there is a canonical isomorphism between $H^*(B\Gamma, \mathbb{C})$ and the group cohomology $H^*(\Gamma, \mathbb{C})$.

The second set of objects determined by the discrete group $\pi_1(M)$ is more analytic in nature. We can consider the reduced C^*-algebra $C^*_r(\Gamma)$, i.e. the closure in $\mathrm{B}(\ell^2(\Gamma))$ of the image of $\mathbb{C}\Gamma$ by the left regular representation, and the infinite dimensional bundles

$$\mathcal{S}^\pm = S^\pm \otimes (\widetilde{M} \times_\Gamma C^*_r(\Gamma)).$$

These are bundles on M with fibres that are finitely generated projective $C^*_r(\Gamma)$-modules. The operator $\widetilde{D}$ defines operators $\mathcal{D}^\pm : C^\infty(M, \mathcal{S}^\pm) \to C^\infty(M, \mathcal{S}^\mp)$ which are $C^*_r(\Gamma)$-Fredholm as maps $H^1(M, \mathcal{S}^\pm) \to L^2(M, \mathcal{S}^\mp)$, in the sense that $[\mathrm{null}(\mathcal{D}^+)] - [\mathrm{null}(\mathcal{D}^-)]$ (really $[\mathrm{null}(\mathcal{D}^+ + \mathcal{R}^+)] - [\mathrm{null}(\mathcal{D}^- + \mathcal{R}^-)] \equiv [\mathcal{L}] - [\mathcal{N}]$ for suitable compact perturbations $\mathcal{R}^\pm$, see [R]) is a formal difference of *finitely generated projective* $C^*_r(\Gamma)$-modules. Thus, as