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LIMITS OF CERTAIN SUBHOMOGENEOUS C^*-ALGEBRAS

Klaus Thomsen

Abstract. — It is shown that the Elliott invariant is a complete invariant for the simple unital C^*-algebras which can be realized as an inductive limit of a sequence of finite direct sums of algebras of the form

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_d, i = 1, 2, \ldots, N\},$$

where $x_1, x_2, \ldots, x_N$ is an arbitrary (finite) set on the circle $\mathbb{T}$ and d is a natural number dividing n. The corresponding range of invariants is identified and the classification result is extended to the non-unital case. A series of results about the structure of these C^*-algebras and the maps between them are also obtained.

Résumé. — On prouve que l'invariant d'Elliott est un invariant complet des C^*-algèbres simples à élément unité qui peuvent être réalisées comme limite inductive d'une suite de sommes finies d'algèbres de la forme

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_d, i = 1, 2, \cdots, N\},$$

où $\{x_1, x_2, \cdots, x_N\} \subset \mathbb{T}$ est un sous-ensemble arbitraire et d un entier divisant n. On détermine l'ensemble des valeurs prises par l'invariant et on étend la classification aux algèbres sans unité. Par ailleurs on donne une série de résultats sur la structure de ces C^*-algèbres.

Contents

INTRODUCTION

Dette arbejde blev færdiggjort i mindet om Birger Iversen

The purpose of this paper is to introduce a new type of building block into the classification of inductive limit C^*-algebras and show that the Elliott invariant is also a complete invariant for the simple unital C^*- algebras which are inductive limits of finite direct sums of these building blocks. The building blocks we consider are of the form

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_d, i = 1, 2, \ldots, N\}$$

where $x_1, x_2, \ldots, x_N$ is an arbitrary finite set of elements on the circle $\mathbb{T}$ and $n, d \in \mathbb{N}$ are natural numbers such that d divides n. Such C^*-algebras will be referred to as *building blocks of type* 2. By taking $d = n$ we just get an ordinary circle algebra, but in general a building block of type 2 will have torsion in its K_1-group. This allows us to introduce torsion in the K_1-group without having more than one kind of building block. This is unlike the approach of Elliott in [**E1**], where torsion was introduced by adding an additional type of building block, the socalled dimension-drop C^*-algebras. Note that the identity map of the dimension-drop algebra $\{f \in C[0,1] \otimes M_n : f(0), f(1) \in M_d\}$ factors through $\{f \in C(\mathbb{T}) \otimes M_n : f(1), f(-1) \in M_d\}$ which is a building block of type 2. Hence an inductive limit of a sequence of finite direct sums of circle algebras and matrix algebras over dimension-drop C^*-algebras is also the limit of a sequence of finite direct sums of building blocks of type 2. Therefore the following theorem, which is our main result, unifies and generalizes the classification result for simple unital inductive limits of finite direct sums of circle algebras, [**E3**], [**NT**], and for simple real rank zero limits of finite direct sums of (matrix algebras over) dimension-drop C^*-algebras in [**E1**], [**DL2**].

THEOREM 0.1. — *Let A and B be simple, unital inductive limits of sequences of finite direct sums of building blocks of type 2. Assume that $\varphi_1 \colon K_1(A) \to K_1(B)$ is an isomorphism, $\varphi_0 \colon K_0(A) \to K_0(B)$ an isomorphism of partially ordered abelian groups with order units and $\varphi_T \colon T(B) \to T(A)$ an affine homeomorphism such that*

$$r_B(\omega)(\varphi_0(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A), \quad \omega \in T(B).$$

It follows that there is a $*$-isomorphism $\varphi \colon A \to B$ such that $\varphi_* = \varphi_1$ on $K_1(A)$, $\varphi_* = \varphi_0$ on $K_0(A)$ and $\varphi^* = \varphi_T$ on $T(B)$.

The maps r_A and r_B in this theorem are the canonical continuous affine surjections from the tracial state space onto the state space of the K_0-group of A and B, respectively.

Let us emphasize one particular consequence of this result. Consider

$$\{f \in C(\mathbb{T}) \otimes M_n : f(1) \in M_d\},$$

which is clearly a building block of type 2. It has exactly the same Elliott invariant as the circle algebra $C(\mathbb{T}) \otimes M_d$, although the algebra seems to be much closer to $C(\mathbb{T}) \otimes M_n$. It would therefore seem tempting to try to use this kind of building blocks to construct two non-isomorphic simple, unital inductive limits of type I C^*-algebras with the same Elliott invariant. This is not possible by the above theorem, in fact a corollary of it says such inductive limits, build only on these very special building blocks of type 2, will automatically be inductive limits of finite direct sums of circle algebras, and hence be subsumed under existing classification results, **[E3]**, **[NT]**. This observation gives some support to the belief that the Elliott invariant will turn out to be a complete invariant for simple inductive limits of more general sub-homogeneous C^*-algebras. It is very challenging to try for such an extension of the existing classification results because even very elementary sub-homogenous C^*-algebras give rise to simple inductive limits which display features that do not arise by using homogeneous building blocks, see **[ET]**, **[Th5]**, **[Th6]**; specifically, the K_0-group can be an arbitrary unperforated simple, (countable) partially ordered abelian group and the restriction map $r_A \colon T(A) \to SK_0(A)$ an arbitrary continuous affine surjection. However, these phenomena do not show up here since we stick to building blocks of type 2. Indeed, if Elliotts conjecture is true, the simple limits we build must also be inductive limits of a sequence of finite direct sums of homogenous C^*- algebras.

In very broad outline, the method of proof we use here is a combination of the methods developed in **[E1]**, **[Th2]**, **[E2]**, **[E3]**, **[DL2]** and **[NT]**. The key words are eigenvalue functions (or characteristic functions as we prefer to call them), determinants, KK-theory and unitary commutators. This paper is the first to handle a case where all these ingredients come into play at the same time. The KK-theory, which is an indispensable ingredient of the classification result in the (non-simple) real rank zero case, **[DL2]**, and the algebraic K_1-group, in the guise of the unitary group modulo the closure of its commutator subgroup, which is needed to determine the approximate inner equivalence class of maps lifted from the Elliott invariant, **[NT]**, play so prominent a role in the development presented here that it almost seems as a miracle that they do not show up in the classification result. They both leave the stage, elegantly we hope, just before the curtain.

On the way we establish several results which are of interest beyond their role in the proof of the classification result. One is that a simple unital inductive limit of a sequence of building blocks of type 2 is approximately divisible (Theorem 5.1), a notion introduced in **[BKR]** and of crucial importance in the previous classification results based on the Elliott invariant which go beyond the real rank zero case, **[E2]**, **[E3]**, **[NT]**. Another important step is the result that two unital $*$-homomorphisms between building blocks of type 2 are approximately inner equivalent when they agree on the tracial states (Theorem 1.4). At first sight it may seem surprising that no K_1-information is needed to reach this conclusion. It shows that exact equality on traces is a strong assumption, although it is of course a necessary condition. The K_1-information first becomes crucial when we consider, as we must, a case where the two maps only agree approximately on the trace level. A third theorem (Theorem B of Chapter 7) gives sufficient (and necessary) conditions for unital $*$-homomorphisms between unital limits of sums of building blocks of type 2 to be approximately inner equivalent when the domain algebra is simple, and we show that a map between the Elliott invariants of the two algebras can be lifted to a $*$-homomorphism when the target algebra is approximately divisible (Corollary A2 of Chapter 7). In fact, we show that the lift can be chosen to be compatible with any KK-element and any map between the unitary groups modulo the closure of their commutator subgroups, which is consistent with the map between the Elliott invariants (Theorem A of Chapter 7).

In the chapters following Chapter 7, which contains the main results, we prove a series of results which relate to the classification result and which are more or less direct consequences of that result and the methods leading to it. In Chapter 8 we describe the quotient group $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ of approximate inner equivalence classes of automorphisms of A when A is a simple unital limit of sums of building blocks of type 2. The main new feature appearing here, when compared with the previous chapters, is the introduction of the quotient $KL(A,A)$ of $KK(A,A)$. By using this device together with some recent results of Dadarlat and Loring, **[DL3]**, we show that $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is the semi-direct product of the group of automorphisms of the Elliott invariant by an abelian group, specifically that

$$\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \simeq$$
$$[\mathrm{ext}(K_1(A), K_0(A)) \oplus \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)] \rtimes \mathrm{Aut}(\mathcal{E}_A).$$

In this expression the third component, $\mathrm{Aut}(\mathcal{E}_A)$, represents the expected part, namely the group of automorphisms of the Elliott invariant. The first component,

$$\mathrm{ext}(K_1(A), K_0(A)),$$

was discovered by Dadarlat and Loring in the real rank zero case, **[DL3]**, in which case the third piece, $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$, is zero (because $\mathrm{Aff}\, T(A) = \overline{\rho(K_0(A))}$). In the case where A is the limit of sums of circle algebras, $\mathrm{ext}(K_1(A), K_0(A))$ is zero, while

$$\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$$

is zero if and only if A has real rank zero or $K_1(A)$ is a torsion group.

In Chapter 9 we describe the range of the Elliott invariant classified by the main result. The range consists of the quadruples (Δ, r, G, H) where Δ is a metrizable Choquet simplex, G is a countable dimension group ($\neq \mathbb{Z}$) with order unit, H a countable abelian group and $r\colon \Delta \to SG$ a continuous affine extreme-point preserving surjection. This characterisation is fairly easily obtained from the work of Villadsen **[V1]**. In order to tie the present work up with previous work dealing with the classification of direct sums of circle algebras and matrix algebras over the dimension-drop C^*-algebras, **[E1]**, **[DL1]** (in the real rank zero case), we show that all the invariants are realized by simple unital inductive limits of sequences of finite direct sums of circle algebras and matrix algebras over dimension drop C^*-algebras. In this way it becomes a corollary of the classification result that any simple unital limit of sums of building blocks of type 2 is also the limit of a sequence of finite direct sums of circle algebras and matrix algebras over dimension-drop C^*-algebras.

In Chapter 10 we show how to extend the classification result to the non-unital case. While this is fairly straightforward and follows the line laid out in **[Th8]**, the other results from the unital case seem more difficult to generalize. In particular, it is not straightforward to describe $\mathrm{Aut}(A)/\overline{\mathrm{Inn}\, A}$ in the non-unital case, and we make no attempts here.

Finally, in Chapter 11 we have gathered a series of consequences of our main results for the structure of the class of C^*-algebras we consider. They all follow fairly straightforwardly by comparing the classification theorem we obtain here with previous work of others, except for the following result which is also of interest for other classes of C^*-algebras. Namely, we prove that the non-stable K-theory is trivial for all unital approximately divisible C^*-algebras, in the sense that the homotopy groups of the unitary group of such a C^*-algebra agree with the K-theory of the algebra, or equivalently, that the unitary group is homotopy equivalent to the 'unitary group' of the stabilized C^*-algebra, see Theorem 11.6.

The first seven chapters of this paper has been circulated in preprint form with the title "Limits of certain subhomogeneous C^*-algebras I".

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