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THE CONVEX AND CONCAVE DECOMPOSITION OF MANIFOLDS WITH REAL PROJECTIVE STRUCTURES

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***Abstract.* —** We try to understand the geometric properties of n-manifolds ($n \geq 2$) with geometric structures modeled on $(\mathbf{R}P^n, \mathrm{PGL}(n+1, \mathbf{R}))$, i.e., n-manifolds with projectively flat torsion free affine connections. We define the notion of i-convexity of such manifolds due to Carrière for integers i, $1 \leq i \leq n-1$, which are generalization of convexity. Given a real projective n-manifold M, we show that the failure of an $(n-1)$-convexity of M implies an existence of a certain geometric object, n-crescent, in the completion $\check{M}$ of the universal cover $\widetilde{M}$ of M. We show that this further implies the existence of a particular type of affine submanifold in M and give a natural decomposition of M into simpler real projective manifolds, some of which are $(n-1)$-convex and others are affine, more specifically concave affine. We feel that it is useful to have such decomposition particularly in dimension three. Our result will later aid us to study the geometric and topological properties of radiant affine 3-manifolds leading to their classification. We get a consequence for affine Lie groups.

***Résumé* (Décomposition convexe et concave des variétés projectives réelles)**

Notre but est de décrire les propriétés projectives réelles géométriques des variétés munies de $(\mathbf{R}P^n, \mathrm{PGL}(n+1, \mathbf{R}))$-structures, où $n \geq 2$, c'est-à-dire des variétés équipées de connexions affines projectivement plates et sans torsion. Nous introduisons la définition de la i-convexité, $1 \leq i \leq n-1$, due à Carrière et généralisant la convexité usuelle. Nous montrons que, si une variété n'est pas $(n-1)$-convexe, alors un certain objet géométrique, appelé i-croissant, existe dans le complété $\check{M}$ du revêtement universel $\widetilde{M}$ de M. De plus, cette dernière propriété entraîne l'existence d'une sous-variété affine d'un certain type dans M et d'une décomposition de M en variétés projectives plus simples, dont certaines sont $(n-1)$-convexes et d'autres affines, plus précisément concaves affines. Une telle décomposition devrait s'avérer utile tout particulièrement en dimension 3. En particulier, nous l'utiliserons pour classifier les variétés affines radiales de dimension 3. Ici nous en déduisons enfin une conséquence pour les groupes de Lie affines.

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PREFACE

The purpose of this monograph is to give a self-contained exposition on recent results in flat real projective structures on manifolds. The main result is that manifolds with such structures have canonical geometric decomposition to manifolds with more structures, i.e., ones with better convexity properties and ones which have affine structures of special types.

We hope that this book exposes some of the newly found materials in real projective structures so that more people might become interested in this topic. For that purpose, we include many details missing from previous papers and try to show that the techniques of this paper are at an elementary level requiring only some visualization in spherical and real projective geometry.

Presently, the global study of real projective structures on manifolds is a field which needs to mature with various relevant tools to be discovered. The fact that such geometric structures do not have metrics and such manifolds are often incomplete creates much confusion. Also, as these structures are often assembled in extremely complicated manner as can be seen from their complicated global charts, or developing maps, the manifolds with such structures cannot be seen as having covers in subsets of model geometric spaces. This means that the arguments must be somewhat delicate.

Let us state some reasons why we are interested in real projective structures: Firstly, our decomposition will be helpful for the study of 3-manifolds with flat real projective structures. Already, the theory helps us in classification of radiant affine 3-manifolds (see [**14**]).

All eight of homogeneous 3-dimensional Riemannian geometries can be seen as manifestations of projective geometries, as observed by Thurston. More precisely, Euclidean, spherical, and hyperbolic geometries have projective models. The same can be said of **Sol**-, **Nil**-, and $\widetilde{\mathrm{SL}}(2, \mathbf{R})$-geometries. $H^2 \times \mathbf{R}^1$- and $\mathbf{S}^2 \times \mathbf{R}^1$-geometries are modeled on $\mathbf{R}P^2 \times \mathbf{R}P^1$. Hence, every 3-manifold with homogeneous Riemannian structure has a natural real projective structure or a product real projective structure.

One could conjecture that many 3-manifolds admit real projective structures although we do not even have a clue how to go about studying such a question.

Classical affine and projective geometries have plethora of beautiful results giving much insight into Euclidean, spherical, and hyperbolic geometries. We expect that such classical theorems will have important roles to play in the global study of projective structures on manifolds although in the present paper only very small portion of classical geometry is ever used.

As we collect more results on various geometric structures on manifolds, we may gain more perspectives on topology of manifolds which are not available from studying relatively better understood Riemannian homogenous geometric structures. By examining more flexible geometric structures such as foliation, symplectic, contact, conformal, affine, or real projective structures, we may gain more informations about the nature of geometric structures and manifolds in general. (We note here that the comparative study of the geometric structures still have not been delved into much.)

The author thanks Bill Thurston who initiated him into this subject which has much beauty, Bill Goldman who had pioneered some early successful results in this field, Yves Carrière who posed many interesting questions with respect to affine structures, and Hyuk Kim for many sharp observations which helped him to think more clearly. The author benefited greatly from conversations with Thierry Barbot, who suggested the words "convex and concave decomposition", Yves Benoist, Richard Bishop, Craig Hodgson, Michael Kapovich, Steven Kerckhoff, Sadayshi Kojima, François Labourie, Kyung Bai Lee, John Millson, and Frank Raymond. The author thanks the Global Analysis Research Center for generous support and allowing him to enjoy doing mathematics at his slow and inefficient pace.

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PART I

AN INTRODUCTION TO REAL PROJECTIVE STRUCTURES

CHAPTER 1

INTRODUCTION

From Ehresmann's definition of geometric structures on manifolds, a *real projective structure* on a manifold is given by a maximal atlas of charts to $\mathbf{R}P^n$ with transition functions extending to projective transformations. (For convenience, we will assume that the dimension n of manifolds is greater that or equal to 2 throughout this paper unless stated otherwise.) This device lifts the real projective geometry locally and consistently on a manifold. In differential geometry, a real projective structure is defined as a projectively flat torsion-free connection. Another equivalent way to define a real projective structure on a manifold M is to give an immersion $\mathbf{dev} : \widetilde{M} \to \mathbf{R}P^n$, a so-called a developing map, equivariant with respect to a so-called holonomy homomorphism $h : \pi_1(M) \to \mathrm{PGL}(n+1, \mathbf{R})$ where $\pi_1(M)$ is the group of deck transformations of the universal cover $\widetilde{M}$ of M and $\mathrm{PGL}(n+1, \mathbf{R})$ is the group of projective transformations of $\mathbf{R}P^n$. (The pair $(\mathbf{dev}, h)$ is said to be the *development pair.*) Each of these descriptions of a real projective structure gives rise to a description of the other two kinds unique up to some natural equivalences.

The global geometric and topological properties of real projective manifolds are completely unknown, and are thought to be very complicated. The study of real projective structure is a fairly obscure area with only handful of global results, as it is a very young field with many open questions, however seemingly unsolvable by traditional methods. The complication comes from the fact that many compact real projective manifolds are not geodesically complete, and often the holonomy groups are far from being discrete lattices and thought to be far from being small such as solvable. There are some early indication that this field however offers many challenges for applying linear representations of discrete groups (which are not lattices), group cohomology, classical convex and projective geometry, affine and projective differential geometry, real algebraic geometry, and analysis. (Since we cannot hope to mention them here appropriately, we offer as a reference the Proceedings of Geometry and Topology Conference at Seoul National University in 1997 **[18]**.) This area is also