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BOUNDARY COHOMOLOGY OF SHIMURA VARIETIES, III: COHERENT COHOMOLOGY ON HIGHER-RANK BOUNDARY STRATA AND APPLICATIONS TO HODGE THEORY

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Abstract. — In this article, third of a series, we complete the verification of the following fact. The nerve spectral sequence for the cohomology of the Borel-Serre boundary of a Shimura variety Sh is a spectral sequence of mixed Hodge–de Rham structures over the field of definition of its canonical model. To achieve that, we develop the machinery of automorphic vector bundles on *mixed* Shimura varieties, for the latter enter in the boundary of the toroidal compactifications of Sh; and study the nerve spectral sequence for the automorphic vector bundles and the toroidal boundary. We also extend the technique of averting issues of base-change by taking cohomology with growth conditions. We give and apply formulas for the Hodge gradation of the cohomology of both Sh and its Borel-Serre boundary.

***Résumé* (Cohomologie au bord des variétés de Shimura, III).** — Dans cet article, troisième d'une série, nous terminons la vérification du fait suivant. La suite spectrale « du nerf », qui calcule la cohomologie du bord de la compactification de Borel-Serre d'une variété de Shimura Sh, est une suite spectrale de structures de Hodge-de Rham mixtes sur le corps de définition de son modèle canonique. Pour le faire, nous développons la théorie de fibrés automorphes sur les variétés de Shimura mixtes, car de tels objets figurent dans le bord d'une compactification toroïdale de Sh ; et nous étudions la suite spectrale « du nerf » pour les fibrés automorphes et le bord toroïdal. En plus, nous généralisons nos résultats antérieurs sur la cohomologie avec conditions de croissance, qui permettent d'éviter les difficultés associées au changement de base. Enfin, nous énonçons et appliquons des formules pour la graduation de Hodge de la cohomologie de Sh et celle du bord de sa compactification de Borel-Serre.

CONTENTS

INTRODUCTION

The present article continues the study of the boundary cohomology of Shimura varieties initiated in [HZ1, HZ2]. Let G be a reductive group over $\mathbb{Q}$, X the symmetric space associated to $G(\mathbb{R})$, and Γ a congruence subgroup of $G(\mathbb{Q})$. We consider the cohomology of $\Gamma\backslash X$ with coefficients in the local system $\widetilde{\boldsymbol{V}}$ constructed from a representation V *of* G, i.e., $H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}}) \simeq H^\bullet(\Gamma, V)$. It is standard that this cohomology can be decomposed as the direct sum of "interior" cohomology, defined as the image of the cohomology with compact supports $H^\bullet_c(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$, and a complementary "boundary cohomology" that restricts non-trivially to the boundary of the Borel-Serre (manifold-with-corners) compactification of $\Gamma\backslash X$. The designation of boundary cohomology is generally non-canonical, and much work has been devoted to constructing canonical decompositions using Eisenstein series.

By an elaboration on the de Rham theorem, one knows that the cohomology group $H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$ can be expressed as the relative Lie algebra cohomology of the space of V-valued C^∞ functions on $\Gamma\backslash G(\mathbb{R})$, or even the functions of moderate growth ([B2, § 7]). Thanks to the work of Franke [Fr1], one can replace the functions of moderate growth by the subspace of automorphic forms, and this can provide the starting point for an approach to the boundary cohomology. However, in this series of articles we are concerned only tangentially with the relation between boundary cohomology and automorphic forms. We choose to work at a more intrinsic level, concentrating instead on the additional structures on $H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$ when X is a hermitian symmetric domain. In that case, $\Gamma\backslash X$ is an algebraic variety, and $\widetilde{\boldsymbol{V}}$ underlies a natural variation of Hodge structure. Morihiko Saito's theory of mixed Hodge modules [Sa3] then gives that $H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$ has a corresponding mixed Hodge structure (MHS). The nature of this MHS at the boundary—more accurately, the associated MHS on the deleted neighborhood cohomology of the boundary—was the subject of [HZ2].

The "adelic version" of $\Gamma\backslash X$ is the Shimura variety $\mathrm{Sh}(G, X)$, whose connected components are of the form $\Gamma\backslash X$. This has a canonical model over a number field E. The de Rham isomorphism identifies $H^\bullet(\mathrm{Sh}(G, X), \widetilde{\boldsymbol{V}})$ with the hypercohomology of of an E-rational complex of coherent sheaves on $\mathrm{Sh}(G, X)$; thus $H^\bullet(\mathrm{Sh}(G, X), \widetilde{\boldsymbol{V}})$

acquires an E-rational structure distinct from the topological rational structure coming from the coefficients $\widetilde{\boldsymbol{V}}$. In particular, $H^\bullet(\mathrm{Sh}(G,X),\widetilde{\boldsymbol{V}})$ has a Hodge filtration whose graded pieces are given by the coherent cohomology with coefficients in certain automorphic vector bundles [H1, Mi2]; the latter have natural E-rational structure. (This E-rationality can be asserted for de Rham cohomology, without grading for the Hodge filtration, and that is conjecturally equivalent in this context.) Study of the boundary cohomology of such automorphic vector bundles was begun in [HZ1].

Both [HZ1] and [HZ2] made essential use of toroidal compactifications of Shimura varieties (which has its origin in [AMRT]), following [H3] and [H4]. The toroidal boundary of $\mathrm{Sh}(G,X)$ (a divisor with normal crossings), like the Borel-Serre boundary, is stratified according to conjugacy classes of parabolic subgroups of G. The cohomology of the boundary, both in the topological setting (as above) and in the coherent setting (i.e., for canonically extended automorphic vector bundles), can be computed as the abutment of the spectral sequence for the closed covering given by this stratification; this is called the *nerve spectral sequence*. In [HZ1] we analyzed the contribution to the nerve spectral sequence from the strata associated to *maximal* parabolics in the coherent setting. The first task of the present article is to extend this analysis to general parabolics, thereby fulfilling our promise from [HZ1], and this is carried out in the first three Chapters. This necessitated a generalization in Chapter 1 of much of the machinery of automorphic vector bundles to the toroidal compactifications of *mixed* Shimura varieties (constructed by Pink [P]).

Most of the calculations from [HZ1, § 3] go over without change, but there are a few delicate points, notably the issue of basechange in (3.4). For the latter, we must recall the role of conditions of growth and decay. These entered in the coherent setting when we established the existence and degeneration of Leray spectral sequences for morphisms of toroidally compactified varieties and the corresponding morphisms of canonical (or subcanonical) extensions of automorphic vector bundles. In effect, it enabled us to circumvent the complications related to basechange at infinity. As suggested above, this last point recurs here. We are obliged to prove (in (2.3)) a generalization to mixed conditions of growth and decay, enabling us, in effect, to isolate a single boundary stratum.

Our main result in Chapter 3 is that the differentials in the E_1-term of the nerve spectral sequence for coherent cohomology decompose naturally into pieces that either are given in terms of restriction maps on pure Shimura varieties or are "purely topological" (see (3.5.4)). Via Franke's interpretation of cohomology in terms of automorphic forms, this implies (see (3.6)) that the constant term maps for cohomology, expressed as integration of an automorphic form along the unipotent radical of appropriate parabolic subgroups, are rational with respect to the de Rham rational structure; for maximal parabolics, this was already obtained in [HZ1, 4.8].

The nerve spectral sequence for the topological cohomology $H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$ was studied in detail in [HZ2]. Hodge-theoretic considerations require (algebraic) compactifications, and the toroidal compactifications were convenient to use for this purpose as well. It was a subtle matter to compare the deleted neighborhoods of the Borel-Serre and toroidal boundary strata associated to a given parabolic subgroup (see [HZ2, § 2]). We constructed isomorphisms between them that are compatible with restriction maps, allowing for transport of structure from the latter to the former. From this, it follows that the differentials in the topological nerve spectral sequence are morphisms of mixed Hodge structures. In particular, they induce maps after grading for the Hodge filtration F. Since a morphism of mixed Hodge structures is determined by its gradation for F, it follows, for instance, that ghost classes exist in $H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$ if and only if they exist in $\mathrm{Gr}_F H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$ (see (4.6.7)). (Recall that a ghost class in $H^\bullet(\Gamma\backslash X, \widetilde{\boldsymbol{V}})$ is a cohomology class whose restriction to the Borel-Serre boundary is non-zero, yet whose restriction to each face (stratum) thereof *is* zero.) In (4.1), we compare the graded differentials to the results obtained for the differentials in the case of the coherent cohomology. To that end, we derive a formula for the deleted neighborhood cohomology of a boundary stratum as de Rham cohomology on a suitable toroidal compactification of the associated (Baily-Borel) boundary component (see (4.1.9)).

Of course, the above can be repeated for the weight filtration. For an example of the use of weights to rule out ghost classes (cf. (4.6.14)), see [Z5, App. A].[1] We are still seeking a satisfactory way of dealing with the entire mixed Hodge structure. It is therefore strongly to be feared that this article is not the last of the series ... The content of the first three chapters of this article completes the verification of results announced in [HZ1, § 5] and in [H5]. They can be summarized by saying that *the (topological) nerve spectral sequence is a spectral sequence of mixed Hodge-de Rham structures over the field of definition of the canonical model.*

In Chapter 4, we continue to develop the Hodge theoretic material from [HZ2, § 5]. In (4.2), we reformulate the results in (4.1) by using the "minimal model" of the holomorphic de Rham complex, viz., the dual Bernstein-Gelfand-Gelfand complex, and deduce the E-rational version of (4.1.9).

A big surprise in this work was the discovery of another interesting filtration on the boundary complex, whose spectral sequence is, like the nerve spectral sequence, a spectral sequence of mixed Hodge structures. In a way, there is nothing new about this filtration, which we call the filtration by holomorphic rank; it is given by the pullback to the Borel-Serre boundary of the filtration of the Baily-Borel Satake boundary by

(1)The correct outcome of the calculation presented in the latter is that there are no ghosts for $GSp(4)$ *when the representation* V *is generic*, i.e., where the highest weight for $Sp(4)$ has positive inner product with both simple roots. When V is trivial, on the other hand, the calculation does allow for a weight-two ghost class in $H^2(\Gamma\backslash X, \mathbb{Q})$, and such a class is determined in [KR, 14.1.3].

(unions of) boundary strata of increasing dimension (see also (4.4.15)). In a sense that can be made precise, its E_1-term is closer to the abutment than that of the nerve spectral sequence, though further from the question of ghost classes. We treat the holomorphic rank filtration in (4.4), though the same considerations already show up in (3.5) in the coherent setting. Cases of the latter give the Hodge components for the E_1-term of the topological holomorphic rank spectral sequence, and this gets examined in (4.5).

Several fundamental questions remain open. The analysis of cohomology of Shimura varieties should be extended to the intersection cohomology of their minimal (Baily-Borel) compactifications. The Zucker conjecture, proved by Looijenga [L] and Saper-Stern [SS], asserts that this cohomology is isomorphic to the L_2-cohomology, or again to the Lie algebra cohomology of square-integrable C^∞ functions, or by [Fr1], of square-integrable automorphic forms. However, it is not known whether this isomorphism identifies Morihiko Saito's Hodge structure with the analytic Hodge structure on L_2-cohomology (the one given a priori by the L^2 harmonic forms). In Chapter 5 we obtain a partial result in this direction: we show that the map from L_2-cohomology of the open Shimura variety to ordinary cohomology is a morphism of (mixed) Hodge structures (this is a small improvement over what was asserted in [H5, 3.3.9]). We do not address the question of whether intersection cohomology carries a de Rham rational structure.

It is also true that not all questions are treated in maximum generality. For instance, we have not studied the cohomology of a general automorphic vector bundle or variation of mixed Hodge structure on a mixed Shimura variety, but have rather been content to work out the cases directly relevant to the cohomology of pure Shimura varieties. Experience suggests these omissions will return to haunt us (providing even more impetus for article IV?). Another thing absent is the exploration of relations between our constructions and the general polylogarithms constructed by Wildeshaus [W1, W2].

Much of this work was begun at the time of writing of [H5], where some of our results were announced. The actual writing of the present article did not get under way until the second-named author visited Université Paris 7 in May, 1997. We both wish to thank that institution for the hospitality extended on that occasion. Likewise, a large amount of the work and writing of this article was carried out while the second-named author was spending Academic Year 1998–99 on sabbatical at the Institute for Advanced Study in Princeton. We also wish to thank P. Polo for helpful discussions of the generalized Bernstein–Bernstein–Gelfand resolution, and Z. Mebkhout for help with the proof of Proposition (4.2.21). We thank J. Wildeshaus for numerous thoughtful comments on both the content and the exposition of the article. Finally, we are grateful to the referee for his careful reading of the first