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WEAKLY RESONANT TUNNELING INTERACTIONS FOR ADIABATIC QUASI-PERIODIC SCHRÖDINGER OPERATORS

Alexander Fedotov, Frédéric Klopp

Abstract. **–** In this paper, we study spectral properties of the one dimensional periodic Schrödinger operator with an adiabatic quasi-periodic perturbation. We show that in certain energy regions the perturbation leads to resonance effects related to the ones observed in the problem of two resonating quantum wells. These effects affect both the geometry and the nature of the spectrum. In particular, they can lead to the intertwining of sequences of intervals containing absolutely continuous spectrum and intervals containing singular spectrum. Moreover, in regions where all of the spectrum is expected to be singular, these effects typically give rise to exponentially small "islands" of absolutely continuous spectrum.

Résumé **(EVet tunnel faiblement résonant pour des opérateurs de Schrödinger quasi-périodiques adiabatiques)**

Cet article est consacré à l'étude du spectre d'une famille d'opérateurs quasi-périodiques obtenus comme perturbations adiabatiques d'un opérateur périodique fixé. Nous montrons que, dans certaines régions d'énergies, la perturbation entraîne des phénomènes de résonance similaires à ceux observés dans le cas de deux puits. Ces effets s'observent autant sur la géométrie du spectre que sur sa nature. En particulier, on peut observer un entrelacement de types spectraux i.e. une alternance entre du spectre singulier et du spectre absolument continu. Un autre phénomène observé est l'apparition d'îlots de spectre absolument continu dans du spectre singulier dus aux résonances.

CONTENTS

CHAPTER 1

INTRODUCTION

The present paper is devoted to the analysis of the family of one-dimensional quasi-periodic Schrödinger operators acting on $L^2(\mathbb{R})$ defined by

$$H_{z,\varepsilon} = -\frac{d^2}{dx^2} + v(x-z) + \alpha\cos(\varepsilon x). \tag{1.1}$$

We assume that

(H1) $v:\ \mathbb{R}\to\mathbb{R}$ is a non constant, locally square integrable, 1-periodic function;
(H2) ε is a small positive number chosen such that $2\pi/\varepsilon$ be irrational;
(H3) z is a real parameter;
(H4) α is a strictly positive parameter that we will keep fixed in most of the paper.

As ε is small, the operator (1.1) is a slow perturbation of the periodic Schrödinger operator

$$H_0 = -\frac{d^2}{dx^2} + v(x) \tag{1.2}$$

acting on $L^2(\mathbb{R})$. To study (1.1), we use the asymptotic method for slow perturbations of one-dimensional periodic equations developed in **[10]** and **[12]**.

The results of the present paper are follow-ups on those obtained in **[11, 14, 13]** for the family (1.1). In these papers, we have seen that the spectral properties of $H_{z,\varepsilon}$ at energy E depend crucially on the position of the *spectral window* $\mathcal{I}(E) := [E-\alpha, E+\alpha]$ with respect to the spectrum of the unperturbed operator H_0. Note that the size of the window is equal to the amplitude of the adiabatic perturbation. In the present paper, the relative position is described in figure 1.1 i.e., we assume that there exists J, an interval of energies, such that, for all $E\in J$, the spectral window $\mathcal{I}(E)$ covers the edges of two neighboring spectral bands of H_0 (see assumption (G)). In this case, one can say that the spectrum in J is determined by the interaction of the neighboring spectral bands induced by the adiabatic perturbation.

The central object of our study is the monodromy equation, a finite difference equation determined by the monodromy matrix for the family (1.1) of almost periodic

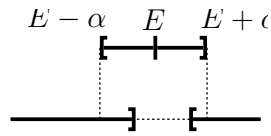

FIGURE 1.1. "Interacting" bands

operators. The monodromy matrix for almost periodic equations with two frequencies was introduced in [**11**]. The passage from (1.1) to the monodromy equation is a non trivial generalization of the monodromization idea from the study of difference equations with periodic coefficients on the real line, see [**6**].

Let us now briefly describe our results and the heuristics underlying them. Let $\mathbf{E}(\kappa)$ be the dispersion relation associated to H_0 (see section 2.1.2); consider the *real* and *complex iso-energy curves*, respectively $\Gamma_{\mathbb{R}}$ and Γ, defined by

$$\Gamma_{\mathbb{R}} := \{(\zeta,\kappa) \in \mathbb{R}^2;\ \mathbf{E}(\kappa) + \alpha \cdot \cos(\zeta) = E\}, \tag{1.3}$$

$$\Gamma := \{(\zeta,\kappa) \in \mathbb{C}^2;\ \mathbf{E}(\kappa) + \alpha \cdot \cos(\zeta) = E\}. \tag{1.4}$$

The dispersion relation $\kappa \mapsto \mathbf{E}(\kappa)$ being multi-valued, in (1.4), we ask that the equation be satisfied at least for one of the possible values of $\mathbf{E}(\kappa)$.

The curves Γ and $\Gamma_{\mathbb{R}}$ are both 2π-periodic in the κ- and ζ-directions; they are described in details in section 11.6. The connected components of $\Gamma_{\mathbb{R}}$ are called *real branches* of Γ.

Consider an interval J such that, for $E \in J$, the assumption on the relative position of the spectral window and the spectrum of H_0 described above is satisfied (see figure 1.1). Then, the curve $\Gamma_{\mathbb{R}}$ consists of an infinite union of connected components, each of which is homeomorphic to a torus; there are exactly two such components in each periodicity cell, see figure 1.2. In this figure, each square represents a periodicity cell. The connected components of $\Gamma_{\mathbb{R}}$ are represented by full lines; we denote two of them by γ_0 and γ_π.

The dashed lines represent loops on Γ that connect certain connected components of $\Gamma_{\mathbb{R}}$; one can distinguish between the "horizontal" loops and the "vertical" loops. There are two special horizontal loops denoted by $\gamma_{h,0}$ and $\gamma_{h,\pi}$; the loop $\gamma_{h,0}$ (resp. $\gamma_{h,\pi}$) connects γ_0 to $\gamma_\pi - (2\pi, 0)$ (resp. γ_0 to γ_π). In the same way, there are two special vertical loops denoted by $\gamma_{v,0}$ and $\gamma_{v,\pi}$; the loop $\gamma_{v,0}$ (resp. $\gamma_{v,\pi}$) connects γ_0 to $\gamma_0 + (0, 2\pi)$ (resp. γ_π to $\gamma_\pi + (0, 2\pi)$).

The standard semi-classical heuristics suggests the following spectral behavior. To each of the loops γ_0 and γ_π, one associates a phase obtained by integrating the fundamental 1-form on Γ along the given loop; let $\Phi_0 = \Phi_0(E)$ (resp. $\Phi_\pi = \Phi_\pi(E)$) be

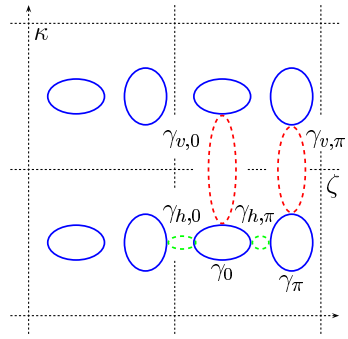

FIGURE 1.2. The adiabatic phase space

one half of the phase corresponding to γ_0 (resp. γ_π). These phases define the quantization conditions:

$$\frac{1}{\varepsilon}\Phi_0(E) = \frac{\pi}{2} + n\pi \qquad \text{and} \qquad \frac{1}{\varepsilon}\Phi_\pi(E) = \frac{\pi}{2} + n\pi, \qquad n \in \mathbb{N}. \tag{1.5}$$

Each of these conditions defines a sequence of energies in J, say $(E_0^{(l)})_l$ and $(E_\pi^{(l')})_{l'}$. For ε sufficiently small, the spectrum of $H_{z,\varepsilon}$ in J should then be located in a neighborhood of these energies.

Moreover, to each of the complex loops $\gamma_{h,0}$, $\gamma_{h,\pi}$, $\gamma_{v,0}$ and $\gamma_{v,\pi}$, one naturally associates an action obtained by integrating the fundamental 1-form on Γ along the loop. For $\nu \in \{0,\pi\}$ and $a \in \{v,h\}$, we denote by $S_{a,\nu}$ the action associated to $\gamma_{a,\nu}$ multiplied by $i/2$. For $E \in \mathbb{R}$, all these actions are real. One orients the loops so that they all be positive. Finally, we define tunneling coefficients as

$$t_{a,\nu} = e^{-S_{a,\nu}/\varepsilon}, \quad \nu \in \{0,\pi\},\ a \in \{v,h\}.$$

When the real iso-energy curve consists in a single torus per periodicity cell (in this case, the energy window overlaps a single edge of a spectral band of H_0 instead of two as in Figure 1.1), the spectrum of $H_{z,\varepsilon}$ is contained in a sequence of intervals described as follows (see [**11**]):

– each interval is neighboring a solution of one of the quantization condition;
– the length of the interval is of the order of the largest tunneling coefficient associated to the loop;
– roughly, the nature of the spectrum is determined by the ratio of the vertical tunneling coefficient to the horizontal one:
  – if this ratio is large, the spectrum is singular;
  – if the ratio is small, the spectrum is absolutely continuous.

In the present case, one must moreover take into account the possible interactions between the tori living in the same periodicity cell. Similarly to what happens in the standard "double well" case (see **[16, 26, 17]**), this effect only plays an important role when the two energies, generated each by one of the tori, are sufficiently close to each other. In this paper, we do not consider the case when these energies are "resonant", i.e. coincide or are "too close" to one another, but we nevertheless "go" up to the case of exponentially close energies.

Let E_0 be an energy satisfying the quantization condition (1.5) defined by Φ_0; let δ be the distance from E_0 to the sequence of energies satisfying the quantization condition (1.5) defined by Φ_π. We now discuss the possible cases depending on this distance. Let us just add that, as the sequences of energies satisfying the quantization equation given by Φ_0 or Φ_π play symmetric roles, in this discussion, the indexes 0 and π can be interchanged freely.

First, we assume that, for some fixed $n > 1$, this distance is of order at least ε^n. In this case, near E_0, the states of the system don't "see" the other lattice of tori, those obtained by translation of the torus γ_π; nor do they "feel" the associated tunneling coefficient $t_{v,\pi}$. Near E_0, everything is as if there was a single torus, namely a translate of γ_0, per periodicity cell. Near E_0, the spectrum of $H_{z,\varepsilon}$ is located in a interval of length of order of the largest of the tunneling coefficients $t_{v,0}$ and $t_h = t_{h,0}t_{h,\pi}$ (see section 2.3.3). And, the nature of the spectrum is determined by quotient $t_{v,0}/t_h$.

So, in the energy region not too close to solutions to both quantization conditions in (1.5), we see that the spectrum is contained in two sequences of exponentially small intervals. For each sequence, the nature of the spectrum is obtained from comparing the vertical to the horizontal tunneling coefficient for the torus generating the sequence. As the tunneling coefficients for both tori are roughly "independent" (see section 2.7.5), it may happen that the spectrum for one of the interval sequences is singular while it is absolutely continuous for the other sequence. If this is the case, one obtains numerous Anderson transitions i.e., thresholds separating a.c. spectrum from singular spectrum (see figure 2.3(b)).

Let us now assume that δ is exponentially small, i.e. of order $e^{-\eta/\varepsilon}$ for some fixed positive η (not too large, see section 2.6). This means that we approach the case of resonant energies. Note that, this implies that there is exactly one energy E_π satisfying (1.5) for Φ_π that is exponentially close to E_0; all other energies satisfying (1.5) for Φ_π are at least at a distance of order ϵ away from E_0.

Then, one can observe two new phenomena. First, there is a repulsion of I_0 and I_π, the intervals corresponding to E_0 and E_π respectively and containing spectrum. This phenomenon is similar to the splitting phenomenon observed in the double well problem (see **[16, 26, 17]**). Second, the interaction can change the nature of the spectrum: the spectrum that would be singular for intervals sufficiently distant from