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INFINITESIMAL ISOPECTRAL DEFORMATIONS OF THE GRASSMANNIAN OF 3-PLANES IN R^6

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INFINITESIMAL ISOSPECTRAL DEFORMATIONS OF THE GRASSMANNIAN OF 3-PLANES IN $\mathbb{R}^6$

Jacques Gasqui, Hubert Goldschmidt

Abstract. — We study the real Grassmannian $G^{\mathbb{R}}_{n,n}$ of n-planes in $\mathbb{R}^{2n}$, with $n \geq 3$, and its reduced space. The latter is the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{n,n}$, which is the quotient of the space $G^{\mathbb{R}}_{n,n}$ under the action of its isometry which sends a n-plane into its orthogonal complement. One of the main results of this monograph asserts that the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{3,3}$ possesses non-trivial infinitesimal isospectral deformations; it provides us with the first example of an irreducible reduced symmetric space which admits such deformations. We also give a criterion for the exactness of a form of degree one on $\bar{G}^{\mathbb{R}}_{n,n}$ in terms of a Radon transform.

Résumé **(Déformations infinitésimales isospectrales de la grassmannienne des 3-plans dans $\mathbb{R}^6$)**

Ce mémoire a pour cadre la grassmannienne $G^{\mathbb{R}}_{n,n}$ des n-plans de $\mathbb{R}^{2n}$, avec $n \geq 3$, et son espace réduit $\bar{G}^{\mathbb{R}}_{n,n}$, qui est l'espace symétrique irréductible, quotient de $G^{\mathbb{R}}_{n,n}$ par l'involution envoyant un n-plan sur son orthogonal. Un de nos principaux résultats est la construction de déformations infinitésimales isospectrales non triviales sur $\bar{G}^{\mathbb{R}}_{3,3}$, obtenant ainsi le premier exemple d'espace symétrique irréductible réduit et non infinitésimalement rigide. Nous donnons aussi un critère d'exactitude pour les formes différentielles de degré 1 sur $\bar{G}^{\mathbb{R}}_{n,n}$, mettant en jeu la nullité d'une transformée de Radon.

CONTENTS

Introduction

We pursue our study of the infinitesimal isospectral deformations of Riemannian symmetric spaces of compact type undertaken in [**5**] and [**6**]. Let (X, g) be a Riemannian symmetric space of compact type. Let $\{g_t\}$ be family of Riemannian metrics on X, with $g_0 = g$. In [**14**], Guillemin proved that, if the family $\{g_t\}$ is an isospectral deformation of g (*i.e.*, if the spectrum of the Laplacian of the metric g_t is independent of t), then the corresponding infinitesimal deformation $h = \frac{d}{dt} g_{t|t=0}$ of the metric g belongs to the kernel $\mathcal{N}_2$ of a certain Radon transform defined on the space of symmetric 2-forms on X in terms of integration over the maximal flat totally geodesic tori of X. The infinitesimal deformation h of g is trivial if it can be written in the form $\frac{d}{dt} \varphi_t^* g_{|t=0}$, where $\{\varphi_t\}$ is one-parameter family of diffeomorphisms of X, or equivalently if it is a Lie derivative of the metric g; such Lie derivatives always belong to the kernel $\mathcal{N}_2$. Consequently, we define the space of infinitesimal isospectral deformations $I(X)$ of X to be the orthogonal complement of the space of Lie derivatives of the metric g in $\mathcal{N}_2$. If the space $I(X)$ vanishes, we say that the space (X, g) is infinitesimally rigid in the sense of Guillemin; under this assumption, an isospectral deformation of the metric g is trivial to first-order and the space X is infinitesimally spectrally rigid (*i.e.*, spectrally rigid to first-order).

The question of Guillemin rigidity for the spaces of rank one first arose in conjunction with the Blaschke problem. The Guillemin rigidity of these spaces which are not spheres was proved by Michel [**19**] for the real projective spaces and by Michel [**19**] and Tsukamoto [**22**] for the other projective spaces (see [**6**]).

The reduced space of X (called the adjoint space in [**6**]), which is constructed in [**16**, Chapter VII], plays a crucial role here; this symmetric space is covered by X and, when X is irreducible, it is not the cover of another symmetric space. We say that X is reduced if it is equal to its reduced space.

We showed that a product of irreducible symmetric spaces is not rigid in the sense of Guillemin (see Theorem 10.5 of [**6**]). Here we prove that an irreducible space which is infinitesimally rigid in the sense of Guillemin must necessarily be reduced (Theorem 1.4). In fact, if X is an irreducible space which is not reduced, then X always possesses an isometry which give rise to symmetric 2-forms which lie in the kernel $\mathcal{N}_2$ of our Radon transform and which are not Lie derivatives of the metric. Thus the relevant problem concerning infinitesimal isospectral deformations for our class of symmetric spaces may be formulated as follows: determine the space of infinitesimal isospectral deformations of an irreducible reduced space.

In [**5**] and [**6**], we began to address this problem for spaces of arbitrary rank and proved that an irreducible symmetric space, which is equal to a Grassmannian, is rigid in the sense of Guillemin if and only if it is reduced. In fact, the Grassmannians $G^{\mathbb{K}}_{m,n}$ of m-planes in $\mathbb{K}^{m+n}$, where $\mathbb{K}$ is a division algebra over $\mathbb{R}$, with $m \neq n$ and $m, n \geq 1$, are rigid. This generalizes the rigidity results for the projective spaces.

We say that a symmetric p-form u on X satisfies the Guillemin condition if, for every maximal flat totally geodesic torus Z contained in X and for all parallel vector fields ζ on Z, the integral

$$\int_Z u(\zeta, \zeta, \ldots, \zeta)\, dZ$$

vanishes, where dZ is the Riemannian measure of Z. The kernel $\mathcal{N}_p$ of the Radon transform for p-forms consists precisely of those forms satisfying the Guillemin condition. Thus the space X is rigid in the sense of Guillemin if and only if the only symmetric 2-forms on X satisfying the Guillemin condition are the Lie derivatives of the metric g. In [**14**], Guillemin proved that a symmetric 2-form, which is equal to the infinitesimal deformation of an isospectral deformation of g, satisfies the Guillemin condition. In [**12**] and [**13**], Grinberg studied the maximal flat Radon transform for functions on an irreducible space. This transform is known to be injective (*i.e.*, the kernel $\mathcal{N}_0$ vanishes) whenever X is one of the irreducible reduced spaces studied in [**6**]. Here we prove the converse of this result for an arbitrary irreducible space: the injectivity of this transform can only occur on a reduced space (Theorem 1.4).

In this monograph, we study the real Grassmannian $G^{\mathbb{R}}_{n,n}$ of n-planes in $\mathbb{R}^{2n}$, with $n \geq 3$, and its reduced space. The latter is the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{n,n}$, which is the quotient of the space $G^{\mathbb{R}}_{n,n}$ under the action of its isometry which sends a plane into its orthogonal complement. The first main result of this monograph asserts that the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{3,3}$ possesses non-trivial infinitesimal isospectral deformations (Theorem 6.2); it provides us with the first example of an irreducible reduced symmetric space which admits such deformations. In fact, we consider an explicit subspace $\mathcal{F}$ of the space of real-valued functions on $\bar{G}^{\mathbb{R}}_{3,3}$ of finite codimension, which is orthogonal to the space of constant functions, and construct an injective mapping

$$\mathcal{F} \longrightarrow I\big(\bar{G}^{\mathbb{R}}_{3,3}\big)$$

which we now describe.

The real Grassmannian $\widetilde{G}^{\mathbb{R}}_{n,n}$ of oriented n-planes in $\mathbb{R}^{2n}$, which is the universal covering manifold of $G^{\mathbb{R}}_{n,n}$, carries a symmetric n-form σ which is invariant under its group of isometries and which is therefore parallel; in fact, the form σ arises from the volume forms of the two canonical bundles of rank n on $\widetilde{G}^{\mathbb{R}}_{n,n}$. This form σ induces a symmetric n-form on $\bar{G}^{\mathbb{R}}_{n,n}$ and an injective mapping $*$ from the space of 1-forms on $\bar{G}^{\mathbb{R}}_{n,n}$ to the space of symmetric $(n-1)$-forms on $\bar{G}^{\mathbb{R}}_{n,n}$. We then show that a 1-form θ on $\bar{G}^{\mathbb{R}}_{n,n}$ satisfies the Guillemin condition if and only if the symmetric $(n-1)$-form $*\theta$ satisfies the Guillemin condition. When $n = 3$, the mapping $*$ sends the space of 1-forms on $\bar{G}^{\mathbb{R}}_{3,3}$ into the space of symmetric 2-forms on $\bar{G}^{\mathbb{R}}_{3,3}$. If f is a real-valued function on $\bar{G}^{\mathbb{R}}_{3,3}$, the symmetric 2-form $*df$ satisfies the Guillemin condition; if f is a non-zero element of $\mathcal{F}$, we prove that the 2-form $*df$ is not a Lie derivative of the metric of $\bar{G}^{\mathbb{R}}_{3,3}$ and thus gives rise to a non-zero element of $I\big(\bar{G}^{\mathbb{R}}_{3,3}\big)$. This construction