TOPICS ON HYPERBOLIC POLYNOMIALS IN ONE VARIABLE

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To my daughter Milena,
to my father
and to the memory of
my mother,
my grandparents,
our friends Despa and Valtchan

TOPICS ON HYPERBOLIC POLYNOMIALS IN ONE VARIABLE

Vladimir Petrov Kostov

Abstract. — The book exposes recent results about hyperbolic polynomials in one real variable, i.e. having all their roots real. It contains a study of the stratification and the geometric properties of the domain in $\mathbb{R}^n$ of the values of the coefficients a_j for which the polynomial $P := x^n + a_1x^{n-1} + \cdots + a_n$ is hyperbolic. Similar studies are performed w.r.t. very hyperbolic polynomials, i.e. hyperbolic and having hyperbolic primitives of any order, and w.r.t. stably hyperbolic ones, i.e. real polynomials of degree n which become hyperbolic after multiplication by x^k and addition of a suitable polynomial of degree $k-1$. New results are presented concerning the Schur-Szegő composition of polynomials, in particular of hyperbolic ones, and of certain entire functions. The question what can be the arrangement of the $\frac{1}{2}n(n+1)$ roots of the polynomials $P, P^{(1)}, \ldots, P^{(n-1)}$ is studied for $n \leq 5$ with the help of the discriminant sets $\mathrm{Res}(P^{(i)}, P^{(j)}) = 0$.

***Résumé* (À propos des polynômes hyperboliques à une variable)**

Le livre expose des résultats récents sur les polynômes hyperboliques (c'est-à-dire à racines réelles) à une variable réelle. Il contient l'étude de la stratification et des propriétés géométriques du domaine dans $\mathbb{R}^n$ des valeurs des coefficients a_j pour lesquelles le polynôme $P := x^n + a_1x^{n-1} + \cdots + a_n$ est hyperbolique. Des études semblables sont effectuées par rapport aux polynômes très hyperboliques, c'est-à-dire hyperboliques et ayant des primitives hyperboliques de tout ordre, et par rapport aux polynômes stablement hyperboliques, c'est-à-dire réels de degré n et qui deviennent hyperboliques après multiplication par x^k et addition d'un polynôme convenable de degré $k-1$. De nouveaux résultats sont présentés qui concernent la composition de Schur-Szegő de polynômes, en particulier hyperboliques, et de certaines fonctions entières. Pour $n \leq 5$, la question « quel peut être l'arrangement des $\frac{1}{2}n(n+1)$ racines des polynômes $P, P^{(1)}, \ldots, P^{(n-1)}$ » est abordée à l'aide des ensembles discriminants $\mathrm{Res}(P^{(i)}, P^{(j)}) = 0$.

CONTENTS

CHAPTER 1

INTRODUCTION

A real polynomial in one variable is *hyperbolic* if it has only real roots. By the Rolle theorem, the derivative of a hyperbolic polynomial of degree $n > 1$ has $n-1$ real roots counted with multiplicity, so it is a hyperbolic polynomial of degree $n-1$. Although being a particular case of real ones, hyperbolic polynomials (HPs) arise in a natural way in many contexts and are interesting to study in their own. Examples are given by characteristic polynomials of symmetric matrices or orthogonal polynomials from the well-known families of Hermite, Laguerre, Chebyshev and Jacobi. (The theory of orthogonal polynomials is exposed in many monographies, for example [**134**] or [**61**].)

In the present book we speak mainly about recent results concerning HPs. (A reader needing a comprehensive book about polynomials could use [**113**], [**125**] or [**126**].) The book contains only part of the proofs of the described results, the ones that illustrate the main ideas and which are worth understanding better. The rest of the proofs can be found in the references.

If one changes continuously the coefficients (hence the roots) of an HP, then this could lead to the appearance of (a) multiple root(s) and subsequent loss of hyperbolicity. To have a good idea about how this happens one needs to study the *hyperbolicity domain* Π of the family of polynomials

$$P := x^n + a_1 x^{n-1} + \cdots + a_n,$$

i.e. the set of values of the real coefficients a_i for which the corresponding polynomial is hyperbolic. This is done in Chapter 2. Most often we set $a_1 = 0$, $a_2 = -1$ (thus decreasing by 2 the number of parameters) which can be obtained by an affine change of the variable x.

In what follows we use the notion of a *multiplicity vector (MV)*, i.e. a vector whose components equal the multiplicities of the distinct roots of an HP when the roots are listed in the increasing order. Sometimes it is better to use a *reverted multiplicity vector (RMV)*, i.e. a vector whose components are the multiplicities of the roots of an HP when the roots are given in the decreasing order.

To understand how multiple roots (and their multiplicity) can appear when the coefficients are varied one has to consider the *stratification* of the hyperbolicity domain (denoted by (S1)) defined by the MVs of the polynomials of the family, see Section 2.1. This is a *Whitney stratification*, i.e. one that satisfies the following two properties:

(a) For two strata A and B such that B is contained in A, denote by T_i the tangent spaces to A at points $x_i \in A$. Suppose that (x_i) is a sequence of points of A converging to some point $y \in B$. Then the limit $\lim_{i\to\infty} T_i$ exists and it contains the tangent space to B at the point y.

(b) With A and B as in property (a), for each sequence $x_1, x_2, \ldots$ of points in A and each sequence $y_1, y_2, \ldots$ of points in B, both converging to the same point y in B, such that the sequence of secant lines L_m between x_m and y_m converges to a line L as $m \to \infty$, and the sequence of tangent planes T_m to A at the points x_m converges to a plane T as $m \to \infty$, one has that L is contained in T. The existence of the limit plane T follows from property (a).

The space $Oa_1 \cdots a_n$ is the one of the coefficients of real monic degree n polynomials. It is natural to study the projections Π^k of the hyperbolicity domain in the subspaces $Oa_1 \cdots a_k$ (i.e. projections which "forget the last $n-k$ coefficients"; these projections arise when a polynomial is being differentiated). This is done in Subsection 2.1.2. It turns out that the non-void fibres of the projections $\Pi^k \to \Pi^{k-1}$ are either segments or points. The latter occur exactly when the point of Π^{k-1} belongs to the boundary of Π^{k-1}, see Theorem 2.1.12.

In Subsection 2.1.4 we cite results of V.I. Arnold, I. Méguerditchian and the author that tell which strata build up the boundary of the projections Π^k. These results are generalized (see Subsection 2.1.5) in the situation when the projections in the spaces $Oa_1 \cdots a_k$ not of Π but of its strata are considered.

Remark 1.0.1. — The results of Chapters 2 and 3 are formulated with the help of reverted multiplicity vectors. In Chapters 4 and 5 we use MVs. The preference for RMVs or MVs is dictated by how it is easier to formulate the results.

Suppose that $a_1 = 0$, $a_2 = -1$. Then the domain Π and the sets Π^k possess the *Whitney property*: The curvilinear distance is equivalent to the Euclidean one. This is proved in Section 2.2. The proof reposes on geometric properties of Π, proved in Subsection 2.1.3, which are of independent interest. The fact that a given compact set has the Whitney property implies that C^k- or C^∞-functions defined on the set can be extended to such functions defined in its neighbourhood.

In Section 2.3 we formulate the result that the non-void *Vandermonde manifolds* are contractible. These manifolds are the preimages of the mapping

$$\text{roots of } P \longmapsto \text{elementary symmetric functions of the roots}$$

when the first l coefficients of the polynomial are fixed. An interesting corollary of their contractibility is that the n ordered values of a random variable are uniquely defined by its first n moments, see Corollary 2.3.5.

Differentiation preserves hyperbolicity, but this is not always the case of integration. That's why in Chapter 3 we consider *very hyperbolic polynomials* of degree n, i.e. hyperbolic and having hyperbolic primitives of all orders, see Section 3.1. We study also *stably hyperbolic* ones, i.e. not necessarily hyperbolic, but which after multiplication by x^k, $k \in \mathbb{N}$, and addition of a suitable degree $k-1$ polynomial become hyperbolic. Stably hyperbolic polynomials are introduced in Section 3.5.

The two classes are closely related. In particular, the domain of stably hyperbolic polynomials of a given degree is obtained from the corresponding domain of very hyperbolic polynomials by linear changes of the coefficients, see Theorem 3.6.6. The domains of very hyperbolic and of stably hyperbolic polynomials of degree 4 and 5 are considered respectively in Sections 3.2 and 3.4.

Remark 1.0.2. — In the present book we speak about strata in several places. In Chapter 3 we describe the boundary of the domain of very hyperbolic polynomials of degree 4 and 5 in terms of strata defined by RMVs some of whose components equal ∞. We do not systematically study the corresponding stratification.

It turns out that the domain of very hyperbolic polynomials has the same properties with regard to the projections in the spaces $Oa_1 \cdots a_k$ (see above) as the domain of hyperbolic polynomials. This is shown in Section 3.3.

The class of diagonal linear operators acting on the space of formal power series and which preserve hyperbolicity of all partial sums of the series is of particular interest to analysts. Such operators are known as multiplier sequences. In Section 3.6 we show how very hyperbolic polynomials arise in the context of multiplier sequences. We also interpret very hyperbolic and stably hyperbolic polynomials in the context of the singularity A_∞ (bearing in mind that hyperbolic polynomials are connected with the singularities of the series A_n; see [**7**] about singularity theory).

It is possible to decide algorithmically whether a given degree 4 polynomial is very (resp. stably) hyperbolic or not. This is proved in Section 3.7. In Subsection 3.7.3 simple necessary conditions for being very or stably hyperbolic are introduced. They are very close to be necessary and sufficient ones. The results of these two sections are illustrated by figures showing the domain of very or stably hyperbolic polynomials of degree 4.

In Chapter 4 we consider the *Schur-Szegő composition.* For two degree n polynomials $P_j := \sum_{i=0}^{n} \binom{n}{i} d_i^j x^i$, $j = 1, 2$, it is defined by the formula

$$P_1 \underset{n}{*} P_2 = \sum_{i=0}^{n} \binom{n}{i} d_i^1 d_i^2 x^i.$$

After introducing the basic definitions and formulae in Section 4.1, we consider in Section 4.2 the case when the polynomials are hyperbolic and one of them has all its roots of the same sign. In this situation the multiplicity vector of $P_1 \underset{n}{*} P_2$ can be deduced from the ones of P_1 and P_2. This class of polynomials is important because Schur-Szegő composition with and only with them preserves hyperbolicity, see Theorem 4.2.2. The sequence $\{d_i\}$ of the coefficients of such a polynomial is called a *finite multiplier sequence.*

It is natural to ask how the roots (and their multiplicities) of $P_1 \underset{n}{*} P_2$ depend on the ones of P_1 and P_2. Or at least what can be said about the numbers of real positive, negative and zero roots of $P_1 \underset{n}{*} P_2$ if these numbers are known for P_1 and P_2. In Section 4.3 we address this question in the case when P_1 and P_2 are hyperbolic (without restriction on the sign of the roots) or when they are arbitrary real polynomials.

In Section 4.4 we show that a given polynomial of the form

$$(x+1)(x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-1})$$

can be presented as a Schur-Szegő composition of $n-1$ polynomials of the form $(x+1)^{n-1}(x+a_i)$. The mapping Φ which sends the tuple $(c_1, \ldots, c_{n-1})$ into the tuple of elementary symmetric polynomials of the quantities a_i is affine and non-degenerate. In Section 4.4.4 we give its eigenvalues (which are rational positive), its eigenvectors (they are defined by HPs) and we formulate its geometric properties. We show that the limit of its jth eigenvector as $n \to \infty$ is expressed via the corresponding Narayana polynomial $(1/j)\sum_{i=1}^{j}\binom{j}{i-1}\binom{j}{i}x^i$ (see Subsection 4.4.3).

Schur-Szegő composition can be defined not only in the class of polynomials but also for entire functions, see Section 4.5. We recall the definition and basic properties of the Laguerre-Pólya class of entire functions in Subsection 4.5.1 (see Definition 4.5.1). This class is of particular interest to us since these functions are uniform limits on compact sets of sequences of hyperbolic polynomials. Subsection 4.5.1 introduces generalizations of Turán's conditions for a function to belong to the Laguerre-Pólya class $\mathcal{LP}$. These generalizations are based on the results about very hyperbolic polynomials exposed in Chapter 3.

The generalization of the mapping Φ for entire functions is considered in Section 4.6. Every entire function of the form $e^x P_n(x)$, where P_n is a degree n polynomial, $P_n(0) = 1$, is expressed as Schur-Szegő composition of n functions of the form $e^x(1 + x/a_i)$. This observation allows to deduce a generalization of the Descartes rule of signs for functions of the form $e^x P_n(x)$, see Theorem 4.6.38.

In Chapter 5 we consider root arrangements for HPs and their derivatives. The classical Rolle theorem states that for an HP, there is a root of its derivative in the interval between any two of its roots. The theorem applies also to the derivatives of an HP (which are also HPs). If the roots of the HP and of its derivatives are written in a string in the increasing order, one obtains the *arrangement* of these roots on the real line. The arrangement is *non-degenerate* if there are no equalities between any two of the roots of the HP and its derivatives. All definitions are given in detail in Section 5.1.1.

The Rolle theorem gives necessary conditions, i.e. restrictions upon these arrangements, and in what follows, we consider only arrangements compatible with the Rolle theorem. It is natural to ask whether these conditions are sufficient as well; that is, whether any arrangement compatible with the Rolle theorem is realizable by the roots of an HP and its derivatives. When the degree of the HP is ≤ 3, the answer to this question is positive.

It turns out that when the degree of the HP is ≥ 4, this is no longer true. The case of degree 4 is considered in detail in Section 5.2 (see Subsection 5.2.1). On Fig. 15 we present the *discriminant sets*, i.e. sets in the space of the coefficients of an HP of degree 4 on which two of the derivatives of the HP have a common root. (The HP is considered to be its 0th derivative.) The figure explains which arrangements are realizable by HPs and their derivatives and which are not.

In that section we introduce a natural generalization of HPs which are the hyperbolic *polynomial-like functions (PLFs)* of degree n, i.e. smooth functions having n roots counted with multiplicity and whose nth derivative vanishes nowhere. (Sometimes they are called functions convex of the order n.) For degree 4 these functions and their derivatives realize all arrangements. Section 5.2 contains the exhaustive answer to the question which degree 4 arrangements are realizable by HPs, which by their perturbations and which by PLFs which are not perturbations of HPs.

Starting with degree 5, the roots of PLFs and their derivatives do not realize all arrangements which are compatible with the Rolle theorem (not even all non-degenerate ones). In Subsection 5.2.2 we consider the discriminant sets for the family of degree 5 HPs; these sets are presented on several figures there. Then we give in Subsection 5.2.3 the exhaustive answer to the question which non-degenerate degree 5 arrangements are realizable by HPs, which by their perturbations, which by PLFs and which are not realizable by PLFs. In Subsection 5.2.4 we explain the reason for 46 of the non-degenerate arrangements not to be realizable by the roots of PLFs and their derivatives (the total number of such arrangements is 50).

Arrangements of the roots of an HP and its derivatives define a stratification (denoted by (S3)). This is also a Whitney stratification. Its strata are subsets of the strata of the stratification (S1) defined by the multiplicity vectors. In Section 5.3 its *overdetermined strata* are considered. They are subsets in the space of the coefficients of the family of HPs on which the arrangement contains a higher number of equalities between roots than expected.

For example in the family of degree 4 HPs $P := x^4 - x^2 + ax + b$ one expects, by varying the two parameters a and b, to obtain two equalities between the roots of the HP and of its derivatives. (It is natural to expect to have one (two etc.) equalities on a codimension one (two etc.) subset in the space of coefficients.) However, when one imposes the condition the polynomial to be divisible by its second derivative (which is equivalent to imposing two equalities between the roots of P and P''), one obtains automatically that the polynomial is even, hence 0 is a root of P' and P''', i.e. there are not two, but three equalities between roots. The section contains examples of overdetermined strata and their exhaustive description up to degree 6.

Remark 1.0.3. — In Subsection 5.1.2 we consider a stratification (S2) of the space $Oa_1 \cdots a_n$ defined by the arrangement of the roots of an HP P and of $P^{(k)}$ for only one value of $k \geq 2$. This is also a Whitney stratification. This stratification shares with (S1) the property that each new equality between roots reduces the dimension of the stratum by 1, and that to each arrangement (to each MV for (S1)) there corresponds a non-empty stratum. This is not the case of (S3). Indeed, there are $\frac{1}{2}n(n+1)$ roots of the HP and all its derivatives whereas there are only n coefficients of the HP. By changing the coefficients one cannot change independently all these roots, therefore one should not expect all arrangements to be realizable by HPs. (And for $n \geq 4$ this is not the case.) On the other hand, overdetermined strata are the ones, where there are more equalities between roots than what is expected from the dimension of the stratum.

In Section 5.4 we consider some problems connected not only with the arrangement of the roots of an HP and its derivatives on the real line, but on the concrete values which these roots can take.

Chapters 2, 3 and 4 of this book should be read in this order. Chapter 5 can be read right after Chapter 2. Chapter 6 contains a survey of other results concerning HPs and the links of some of them to the rest of this book.

In the present book we use the following abbreviations:
HP – *hyperbolic polynomial*, see the beginning of Section 2.1;
(R)MV – *(reverted) multiplicity vector*, see Definition 2.1.1;
CV – *configuration vector* (defined after Example 5.1.2);
FMS$(n+1)$ – *finite multiplier sequence of length* $n+1$ (see Definition 4.2.1);
PLF – *polynomial-like function* (see the beginning of Section 5.1.1);
TAT – *triple of admissible triples* (see Definition 4.3.14);
SSC – *Schur-Szegő composition* (see the beginning of Section 4.1).

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