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AFFINE BRAID GROUP ACTIONS ON DERIVED CATEGORIES OF SPRINGER RESOLUTIONS

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ABSTRACT. – In this paper we construct and study an action of the affine braid group associated with a semi-simple algebraic group on derived categories of coherent sheaves on various varieties related to the Springer resolution of the nilpotent cone. In particular, we describe explicitly the action of the Artin braid group. This action is a "categorical version" of Kazhdan-Lusztig-Ginzburg's construction of the affine Hecke algebra, and is used in particular by the first author and I. Mirković in the course of the proof of Lusztig's conjectures on equivariant K-theory of Springer fibers.

RÉSUMÉ. – Dans cet article nous construisons et étudions une action du groupe de tresses affine associé à un groupe algébrique semi-simple sur les catégories dérivées de faisceaux cohérents sur diverses variétés liées à la résolution de Springer du cône nilpotent. En particulier, nous décrivons explicitement l'action du groupe de tresses d'Artin. Cette action est une « version catégorique » de la construction géométrique de l'algèbre de Hecke affine due à Kazhdan-Lusztig et Ginzburg, et est utilisée par le premier auteur et I. Mirković au cours de la preuve des conjectures de Lusztig sur la K-théorie équivariante des fibres de Springer.

Introduction

0.1. – The goal of this paper is to introduce an action of the affine braid group on the derived category of coherent sheaves on the Springer resolution (and some related varieties) and prove some of its properties.

The most direct way to motivate this construction is via the well-known heuristics of Springer correspondence theory. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$,[1] let $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the Grothendieck-Springer map and $\pi' : \widetilde{\mathcal{N}} \to \mathcal{N} \subset \mathfrak{g}$ be the Springer map; here $\mathcal{N}$ is the nilpotent cone and $\widetilde{\mathcal{N}}$ is the cotangent bundle to the flag variety. Let $\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$ be the subset of regular elements and $\widetilde{\mathfrak{g}}_{\mathrm{reg}} = \pi^{-1}(\mathfrak{g}_{\mathrm{reg}})$. Then $\pi|_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}}$ is a ramified Galois covering with Galois group W, the Weyl group of $\mathfrak{g}$. Thus W acts on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ by deck transformations. Although the

(1) In the body of the paper we work over a finite localization of $\mathbb{Z}$ or over a field of arbitrary characteristic rather than over $\mathbb{C}$. Such details are ignored in the introduction.

action does not extend to an action on $\widetilde{\mathfrak{g}}$, it still induces various interesting structures on the Springer resolution $\widetilde{\mathcal{N}}$. The most well-known example is the Springer action of W on (co)homology of a fiber of π', called a Springer fiber. The procedure of passing from the action of W on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ to the Springer action can be performed using different tools, such as minimal (Goresky-MacPherson) extension of a perverse sheaf (see [47] for the original idea of this construction, and [40] for a detailed treatment and further references), nearby cycles (see [31]), or degeneration of the correspondence cycle (see [28, 22]).

The main result of this paper can also be viewed as a realization of that general idea. Namely, we show that a "degeneration" of the action of W on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ provides an action of the corresponding Artin braid group $\mathbb{B}$ on the derived categories of coherent sheaves $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$, $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$. More precisely, we consider the closure Z_w of the graph of the action of $w \in W$ on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$. Using this as a correspondence we get a functor $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}) \to \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$; we then prove that there exists an action of $\mathbb{B}$ on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ where a minimal length representative $T_w \in \mathbb{B}$ of $w \in W$ acts by the resulting functor. It also induces a compatible action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$.

The fact that functors admitting such a simple description give an action of the braid group is perhaps surprising; it implies that the closures of the graphs are Cohen-Macaulay.

Furthermore, the categories in question carry an obvious action of the weight lattice $\mathbb{X}$ of G which is identified with the Picard group of the flag variety; here an element $\lambda \in \mathbb{X}$ acts by twist by the corresponding line bundle. We prove that this action of $\mathbb{X}$ together with the above action of $\mathbb{B}$ generate an action of the *extended affine braid group*[2] $\mathbb{B}_{\mathrm{aff}}$.

In fact we construct a structure stronger than just an action of $\mathbb{B}_{\mathrm{aff}}$ on the two derived categories of coherent sheaves; namely, we show the existence of a (weak) *geometric action* of this group. Informally, this means that the action of elements of the group come from "integral kernels," i.e. complexes of sheaves on the square of the space, and relations of the group come from isomorphisms between convolutions of the kernels. Formal definition of this convolution requires basic formalism of differential graded schemes. On the other hand, this geometric action induces a usual action on the derived categories of varieties obtained from $\widetilde{\mathcal{N}}$, $\widetilde{\mathfrak{g}}$ by base change. In the simplest case of base change to the transversal slice to a subregular nilpotent orbit we recover the action of $\mathbb{B}_{\mathrm{aff}}$ on the derived category of the minimal resolution of a Kleinian singularity considered e.g. in [57, 17].

Here the term "weak" indicates that our kernels satisfy the relations in $\mathbb{B}_{\mathrm{aff}}$ only up to isomorphism. There is a stronger notion of group action on a category (see [24]) in which some compatibility conditions are imposed on these isomorphisms. We do not consider such a notion in this paper. Let us mention however that it follows from our results that the action of the Artin braid group $\mathbb{B}$ can be endowed with such a structure, see Remark 2.2.2(4).

[2] In the standard terminology (see [15]) this is the extended affine braid group of the *Langlands dual* group ${}^LG$. In fact, this may be viewed as the simplest manifestation of the relation of our $\mathbb{B}_{\mathrm{aff}}$-action to Langlands duality mentioned below.

0.2. – We now list some contexts where the action of $\mathbb{B}_{\mathrm{aff}}$ appears and plays an essential role (see [9] for a complementary discussion).

The work [11] uses the geometric theory of representations of semi-simple Lie algebras in positive characteristic developed in [13, 12] to deduce Lusztig's conjectures on numerical properties of such representations (see [48, 49]). It uses the action considered in this paper, which is related to the action of $\mathbb{B}_{\mathrm{aff}}$ on the derived category of modular representations by *intertwining* functors (or shuffling functors, or Radon transforms, in a different terminology). In fact, [11] and the present paper are logically interdependent. This application was our main motivation for considering this action over a localization of $\mathbb{Z}$. The action studied in this paper (and in particular its version for certain differential graded schemes) also plays a technical role in the study of Koszul duality for representations of semi-simple Lie algebras in positive characteristic (see [55]).

The induced action of $\mathbb{B}_{\mathrm{aff}}$ on the Grothendieck group of $\mathbb{C}^*$-equivariant coherent sheaves factors through an action of the *affine Hecke algebra*, i.e. the action of simple reflections satisfies a certain quadratic relation. A weak form of the categorical counterpart of the quadratic relation is used in [11]; a more comprehensive development of the idea that our action induces an action of the "categorical affine Hecke algebra" is the subject of [25].

In fact, in view of the work of Lusztig and Ginzburg, the monoidal category $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{C}^*}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})$ (or the $\widetilde{\mathcal{N}}$-version; the monoidal structure on these categories is defined below) can be considered as a categorification of the affine Hecke algebra; see [10] and announcement in [9] for an equivalence between this categorification and another one coming from perverse sheaves on the affine flag variety of the dual group. Such an equivalence, inspired by the ideas of local geometric Langlands duality theory, also implies the existence of the $\mathbb{B}_{\mathrm{aff}}$-action constructed in this paper (at least over $\mathbb{C}$).

Another approach to the construction of the $\mathbb{B}$-action (over $\mathbb{C}$) relates it to the well-known action of $\mathbb{B}$ on the category of D-modules on the flag variety by Radon transforms (see e.g. [5]). Passing from D-modules to coherent sheaves on the cotangent bundle is achieved by means of the Hodge D-modules formalism. We plan to develop this approach in a future publication.

Finally, we would like to mention that in the $\widetilde{\mathcal{N}}$-version of the construction, (inverses of) simple generators act by reflection at a spherical functor in the sense of [2, 56] (see Remark 1.6.2), and that in the particular case of groups of type **A** the action (in its non-geometric form, and over $\mathbb{C}$ rather than a localization of $\mathbb{Z}$) has been constructed in [45] and more recently, as a part of a more general picture, in [21].

0.3. Contents of the paper

In Section 1 we prove that there exists an action of $\mathbb{B}_{\mathrm{aff}}$ on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ and $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$ where generators associated with *simple reflections* in W and elements of $\mathbb{X}$ act as stated above. This result was already proved under stronger assumptions and by less satisfactory methods in [54]. We also extend this result to the schemes over a finite localization of $\mathbb{Z}$.

In Section 2 we prove that, if p is bigger than the Coxeter number of G, the action of the element $T_w \in \mathbb{B}$ ($w \in W$) is the convolution with kernel $\mathcal{O}_{Z_w}$. This proof is based on representation theory of semi-simple Lie algebras in positive characteristic. We also extend this result to the schemes over a finite localization of $\mathbb{Z}$, and (as an immediate consequence) over an algebraically closed field of characteristic zero.

In Section 3 we prove generalities on dg-schemes, extending results of [55, §1]. (Here, we concentrate on *quasi-coherent* sheaves.) In particular, we prove a projection formula and a (non-flat) base change theorem in this context.

In Section 4 we use the results of Section 3 to show that the action of $\mathbb{B}_{\mathrm{aff}}$ induces actions on categories of coherent sheaves on various (dg-)varieties related to $\widetilde{\mathscr{N}}$ and $\widetilde{\mathfrak{g}}$, in particular inverse images of Slodowy slices under the Springer resolution.

Finally, in Section 5 we prove some equivariant analogues of the results of Section 4 which are needed in [11].

0.4. Acknowledgements

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1. Existence of the action

1.1. Notation

Let $G_{\mathbb{Z}}$ be a split connected, simply-connected, semi-simple algebraic group over $\mathbb{Z}$. Let $T_{\mathbb{Z}} \subset B_{\mathbb{Z}} \subset G_{\mathbb{Z}}$ be a maximal torus and a Borel subgroup in $G_{\mathbb{Z}}$. Let $\mathfrak{t}_{\mathbb{Z}} \subset \mathfrak{b}_{\mathbb{Z}} \subset \mathfrak{g}_{\mathbb{Z}}$ be their respective Lie algebras. Let also $U_{\mathbb{Z}}$ be the unipotent radical of $B_{\mathbb{Z}}$, and $\mathfrak{n}_{\mathbb{Z}}$ be its Lie algebra. Let Φ be the root system of $(G_{\mathbb{Z}}, T_{\mathbb{Z}})$, and Φ^+ be the positive roots, chosen as the roots of $\mathfrak{g}_{\mathbb{Z}}/\mathfrak{b}_{\mathbb{Z}}$. Let Σ be the associated system of simple roots. Let also $\mathbb{X} := X^*(T_{\mathbb{Z}})$ be the weight lattice. We let $\mathfrak{g}_{\mathbb{Z}}^*$ be the coadjoint representation of $G_{\mathbb{Z}}$.

Let W be the Weyl group of Φ, and let $\mathscr{S} = \{s_\alpha,\ \alpha \in \Sigma\}$ be the set of Coxeter generators associated with Σ (called *simple reflections*). Let $W_{\mathrm{aff}}^{\mathrm{Cox}} := W \ltimes \mathbb{Z}R$ be the affine Weyl group, and $W_{\mathrm{aff}} := W \ltimes \mathbb{X}$ be the extended affine Weyl group. Let $\mathbb{B} \subset \mathbb{B}_{\mathrm{aff}}^{\mathrm{Cox}} \subset \mathbb{B}_{\mathrm{aff}}$ be the braid groups associated with $W \subset W_{\mathrm{aff}}^{\mathrm{Cox}} \subset W_{\mathrm{aff}}$ (see e.g. [12, §2.1.1] or [54, §1.1]). Note that W and $W_{\mathrm{aff}}^{\mathrm{Cox}}$ are Coxeter groups, but not W_{aff} in general. For $s, t \in \mathscr{S}$, let us denote by $n_{s,t}$ the order of st in W. Recall (see [14] or [50, § 3.3]) that $\mathbb{B}_{\mathrm{aff}}$ has a presentation with generators $\{T_s,\ s \in \mathscr{S}\}$ and $\{\theta_x,\ x \in \mathbb{X}\}$ and the following relations:

(i) $T_s T_t \cdots = T_t T_s \cdots$ ($n_{s,t}$ elements on each side);

(ii) $\theta_x \theta_y = \theta_{x+y}$;

(iii) $T_s \theta_x = \theta_x T_s$ if $s(x) = x$;

(iv) $\theta_x = T_s \theta_{x-\alpha} T_s$ if $s = s_\alpha$ and $s(x) = x - \alpha$.

Relations of type (i) are called *finite braid relations*.