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ASYMPTOTIC STABILITY IN THE ENERGY SPACE FOR DARK SOLITONS OF THE GROSS-PITAEVSKII EQUATION

BY Fabrice BÉTHUEL, Philippe GRAVEJAT AND Didier SMETS

ABSTRACT. – We pursue our work [5] on the dynamical stability of dark solitons for the one-dimensional Gross-Pitaevskii equation. In this paper, we prove their asymptotic stability under small perturbations in the energy space. In particular, our results do not require smallness in some weighted spaces or a priori spectral assumptions. Our strategy is reminiscent of the one used by Martel and Merle in various works regarding generalized Korteweg-de Vries equations. The important feature of our contribution is related to the fact that while Korteweg-de Vries equations possess unidirectional dispersion, Schrödinger equations do not.

RÉSUMÉ. – Nous poursuivons notre analyse [5] de la stabilité dynamique des solitons sombres pour l'équation de Gross-Pitaevskii en dimension un. Dans cet article, nous démontrons leur stabilité asymptotique par rapport à de petites perturbations dans l'espace d'énergie. En particulier, nos résultats ne requièrent aucune condition de petitesse dans des espaces à poids, aussi bien qu'aucune hypothèse spectrale *a priori*. Notre stratégie s'appuie sur celle développée par Martel et Merle dans plusieurs articles au sujet des équations de Korteweg-de Vries généralisées. Notre contribution principale réside dans le fait que les équations de Korteweg-de Vries possèdent une dispersion unidirectionnelle, ce qui n'est plus le cas des équations de Schrödinger.

1. Introduction

We consider the one-dimensional Gross-Pitaevskii equation

$$i\partial_t \Psi + \partial_{xx}\Psi + \Psi\big(1 - |\Psi|^2\big) = 0, \tag{GP}$$

for a function $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, supplemented with the boundary condition at infinity

$$|\Psi(x,t)| \to 1, \quad \text{as} \quad |x| \to +\infty. \tag{1}$$

The three-dimensional version of (GP) was introduced in the context of Bose-Einstein condensation in [37, 24]. It is also used as a model in other areas of physics such as nonlinear optics [25] and quantum fluid mechanics [14]. In nonlinear optics, the Gross-Pitaevskii equation appears as an envelope equation in optical fibers, and is mostly relevant in the one and two dimensional cases. In dimension one, the case studied in this paper, it models the propagation of dark pulses in slab waveguides, and the boundary condition (1) corresponds to a non-zero background.

On a mathematical level, the Gross-Pitaevskii equation is a defocusing nonlinear Schrödinger equation. It is Hamiltonian, and in dimension one, it owns the remarkable property to be integrable by means of the inverse scattering method [42]. The Hamiltonian is given by the Ginzburg-Landau energy

$$\mathcal{E}(\Psi) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x \Psi|^2 + \frac{1}{4}\int_{\mathbb{R}} (1 - |\Psi|^2)^2.$$

A soliton with speed c is a travelling-wave solution of (GP) of the form

$$\Psi(x, t) := U_c(x - ct).$$

Its profile U_c is a solution to the ordinary differential equation

$$-ic\partial_x U_c + \partial_{xx} U_c + U_c\big(1 - |U_c|^2\big) = 0. \tag{2}$$

The solutions to (2) with finite Ginzburg-Landau energy are explicitly known. For $|c| \geq \sqrt{2}$, they are the constant functions of unitary modulus, while for $|c| < \sqrt{2}$, up to the invariances of the problem, i.e., multiplication by a constant of modulus one and translation, they are uniquely given by the expression

$$U_c(x) := \sqrt{\frac{2 - c^2}{2}}\,\mathrm{th}\Big(\frac{\sqrt{2 - c^2}x}{2}\Big) + i\frac{c}{\sqrt{2}}. \tag{3}$$

Notice that solitons U_c with speed $c \neq 0$ do not vanish on $\mathbb{R}$. They are called dark solitons, with reference to nonlinear optics where $|\Psi|^2$ refers to the intensity of light. Instead, since it vanishes at one point, U_0 is called the black soliton. Notice also, this turns out to be an important feature, that solitons U_c with $c \simeq \sqrt{2}$ have indefinitely small energy.

Our goal in this paper is to study the (GP) flow for initial data that are close to dark solitons, and in particular to analyze the stability of solitons. Since we deal with an infinite dimensional dynamical system, the notion of stability relies heavily on the way to measure distances. A preliminary step is to address the Cauchy problem with respect to these distances. In view of the Hamiltonian $\mathcal{E}$, the natural energy space for (GP) is given by

$$\mathcal{X}(\mathbb{R}) := \big\{\Psi \in H^1_{\rm loc}(\mathbb{R}),\ \Psi' \in L^2(\mathbb{R}) \text{ and } 1 - |\Psi|^2 \in L^2(\mathbb{R})\big\}.$$

Due to the non-vanishing conditions at infinity, it is not a vector space. Yet $\mathcal{X}(\mathbb{R})$ can be given a structure of complete metric space through the distance

$$d(\Psi_1, \Psi_2) := \big\|\Psi_1 - \Psi_2\big\|_{L^\infty([-1,1])} + \big\|\Psi_1' - \Psi_2'\big\|_{L^2(\mathbb{R})} + \big\||\Psi_1| - |\Psi_2|\big\|_{L^2(\mathbb{R})}.$$

In space dimension one, for an initial datum $\Psi^0 \in \mathcal{X}(\mathbb{R})$, the Gross-Pitaevskii equation possesses a unique global solution $\Psi \in \mathcal{C}^0(\mathbb{R}, \mathcal{X}(\mathbb{R}))$, and moreover $\Psi - \Psi^0 \in \mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}))$ (see e.g., [43, 19, 21] and Appendix A.1). In the sequel, stability is based on the distance d.

It is well-known that a perturbation of a soliton $U_{\mathfrak{c}}$ at initial time cannot remain a perturbation of the same soliton for all time. This is related to the fact that there is a continuum of solitons with different speeds. If $\Psi^0 = U_c$ for $c \simeq \mathfrak{c}$, but $c \neq \mathfrak{c}$, then $\Psi(x, t) = U_c(x - ct)$ diverges from $U_{\mathfrak{c}}(x - \mathfrak{c}t)$, as $t \to +\infty$. The notion of orbital stability is tailored to deal with such situations. It means that a solution corresponding to a perturbation of a soliton $U_{\mathfrak{c}}$ at initial time remains a perturbation of the family of solitons with same speed for all time. Orbital stability of dark and black solitons was proved in [26, 3] (see also [1, 22, 5]).

THEOREM 1 ([26, 3]). – *Let* $\mathfrak{c} \in (-\sqrt{2}, \sqrt{2})$. *Given any positive number* ε, *there exists a positive number* δ *such that, if*

$$d\big(\Psi^0, U_{\mathfrak{c}}\big) \leq \delta,$$

then

$$\sup_{t\in\mathbb{R}} \inf_{(a,\theta)\in\mathbb{R}^2} d\big(\Psi(\cdot,t), e^{i\theta} U_{\mathfrak{c}}(\cdot - a)\big) \leq \varepsilon.$$

The proof of Theorem 1 is mainly variational. Following the strategy developed in [11] or [41, 23], it combines minimizing properties of the solitons with the Hamiltonian nature of (GP) through conservation of energy and momentum.

In the sequel, our focus is put on the notion of asymptotic stability. For a finite dimensional system, asymptotic stability of a stationary state or orbit means that any small perturbation of the given state at initial time eventually converges to that state as time goes to infinity. For a finite dimensional Hamiltonian system, this is excluded by the symplectic structure. In infinite dimension, one may take advantage of different topologies to define relevant notions of asymptotic stability. Our main result is

THEOREM 2. – *Let* $\mathfrak{c} \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$. *There exists a positive number* $\delta_{\mathfrak{c}}$, *depending only on* $\mathfrak{c}$, *such that, if*

$$d\big(\Psi^0, U_{\mathfrak{c}}\big) \leq \delta_{\mathfrak{c}},$$

then there exist a number $\mathfrak{c}^* \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$, *and two functions* $b \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ *and* $\theta \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ *such that*

$$b'(t) \to \mathfrak{c}^*, \quad \text{and} \quad \theta'(t) \to 0,$$

as $t \to +\infty$, *and for which we have*

(4)
$$e^{-i\theta(t)} \Psi\big(\cdot + b(t), t\big) \to U_{\mathfrak{c}^*} \quad \text{in } L^\infty_{\rm loc}(\mathbb{R}), \quad \text{and} \quad e^{-i\theta(t)} \partial_x \Psi\big(\cdot + b(t), t\big) \rightharpoonup \partial_x U_{\mathfrak{c}^*} \quad \text{in } L^2(\mathbb{R}),$$

in the limit $t \to +\infty$.

Whereas Theorem 1 establishes that the solution remains close to the whole family of dark solitons, Theorem 2 describes a convergence to some orbit on that family. In particular, it expresses the fact that in the reference frame of the limit soliton, the perturbation is dispersed towards infinity.

Concerning the topology, the convergence in $L^\infty_{\rm loc}(\mathbb{R})$ in (4) cannot be improved into a convergence in $L^\infty(\mathbb{R})$, for instance due to the presence of additional small solitons, or to a possible phenomenon of slow phase winding at infinity. Similarly, the weak convergence of the gradients in $L^2(\mathbb{R})$ cannot be improved into a strong convergence in $L^2(\mathbb{R})$, due to the Hamiltonian nature of the equation. Yet, it is not impossible that the latter could be improved into a strong convergence in $L^2_{\rm loc}(\mathbb{R})$, but we have no proof of that fact.