ISSN 0012-9593

quatrième série - tome 49 fascicule 2 mars-avril 2016

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Pramod N. ACHAR & Simon RICHE

Modular perverse sheaves on flag varieties I: tilting and parity sheaves

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : `smf@smf.univ-mrs.fr`

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 49, 2016, p. 325 à 370

MODULAR PERVERSE SHEAVES ON FLAG VARIETIES I: TILTING AND PARITY SHEAVES

BY PRAMOD N. ACHAR AND SIMON RICHE
WITH A JOINT APPENDIX WITH GEORDIE WILLIAMSON

ABSTRACT. – In this paper we prove that the category of parity complexes on the flag variety of a complex connected reductive group G is a "graded version" of the category of tilting perverse sheaves on the flag variety of the dual group $\check{G}$, for any field of coefficients whose characteristic is good for G. We derive some consequences on Soergel's modular category $\mathcal{O}$, and on multiplicities and decomposition numbers in the category of perverse sheaves.

RÉSUMÉ. – Dans cet article nous démontrons que la catégorie des complexes à parité sur la variété de drapeaux d'un groupe réductif complexe connexe G est une « version graduée » de la catégorie des faisceaux pervers basculants sur la variété de drapeaux du groupe dual $\check{G}$, pour tout corps de coefficients dont la caractéristique est bonne pour G. Nous en déduisons des conséquences sur la catégorie $\mathcal{O}$ modulaire de Soergel, et sur le calcul des multiplicités et des nombres de décomposition dans la catégorie des faisceaux pervers.

1. Introduction

1.1. – This paper is the first in a series devoted to investigating the structure of the category of Bruhat-constructible perverse sheaves on the flag variety of a complex connected reductive algebraic group, with coefficients in a field of positive characteristic. In this part, adapting some constructions of Bezrukavnikov-Yun [14] in the characteristic-0 setting, we show that in good characteristic, the category of parity sheaves on the flag variety of a reductive group is a "graded version" of the category of tilting perverse sheaves on the flag variety of the Langlands dual group. We also derive a number of interesting consequences of this result, in particular on the computation of multiplicities of simple perverse sheaves in standard perverse sheaves, on Soergel's "modular category $\mathcal{O}$," and on decomposition numbers.

P.A. was supported by NSF Grant No. DMS-1001594. S.R. was supported by ANR Grants No. ANR-09-JCJC-0102-01, ANR-2010-BLAN-110-02 and ANR-13-BS01-0001-01.

1.2. Some notation

Let G be a complex connected reductive algebraic group, and let $T \subset B \subset G$ be a maximal torus and a Borel subgroup. The choice of B determines a choice of positive roots of (G, T), namely those appearing in $\mathrm{Lie}(B)$. Consider also the Langlands dual data $\check{T} \subset \check{B} \subset \check{G}$. That is, $\check{G}$ is a complex connected reductive group, and we are given an isomorphism $X^*(T) \cong X_*(\check{T})$ which identifies the roots of G with the coroots of $\check{G}$ (and the positive roots determined by B with the positive coroots determined by $\check{B}$).

We are interested in the varieties $\mathscr{B} := G/B$ and $\check{\mathscr{B}} := \check{G}/\check{B}$, in the derived categories

$$D^{\mathrm{b}}_{(B)}(\mathscr{B}, \Bbbk), \qquad \text{resp.} \qquad D^{\mathrm{b}}_{(\check{B})}(\check{\mathscr{B}}, \Bbbk)$$

of sheaves of $\Bbbk$-vector spaces on these varieties, constructible with respect to the stratification by B-orbits, resp. $\check{B}$-orbits (where $\Bbbk$ is a field), and in their abelian subcategories

$$\mathsf{P}_{(B)}(\mathscr{B}, \Bbbk), \qquad \text{resp.} \qquad \mathsf{P}_{(\check{B})}(\check{\mathscr{B}}, \Bbbk)$$

of perverse sheaves (for the middle perversity). The category $\mathsf{P}_{(B)}(\mathscr{B}, \Bbbk)$ is highest weight, with simple objects $\{\mathscr{IC}_w,\ w \in W\}$, standard objects $\{\Delta_w,\ w \in W\}$, costandard objects $\{\nabla_w,\ w \in W\}$, indecomposable projective objects $\{\mathscr{P}_w,\ w \in W\}$ and indecomposable tilting objects $\{\mathscr{T}_w,\ w \in W\}$ naturally parametrized by the Weyl group W of (G, T). Similar remarks apply of course to $\mathsf{P}_{(\check{B})}(\check{\mathscr{B}}, \Bbbk)$, and we denote the corresponding objects by $\check{\mathscr{IC}}_w$, $\check{\Delta}_w$, $\check{\nabla}_w$, $\check{\mathscr{P}}_w$, $\check{\mathscr{T}}_w$. (Note that the Weyl group of $(\check{G}, \check{T})$ is canonically identified with W.)

1.3. The case $\Bbbk = \mathbb{C}$

These categories have been extensively studied in the case $\Bbbk = \mathbb{C}$: see in particular [10, 9, 14]. To state some of their properties we need some notation. We will denote by $\mathsf{IC}_{(\check{B})}(\check{\mathscr{B}}, \mathbb{C})$ the additive category of semisimple objects in $D^{\mathrm{b}}_{(\check{B})}(\check{\mathscr{B}}, \mathbb{C})$ (i.e., the full subcategory whose objects are direct sums of shifts of simple perverse sheaves). If A is an abelian category, we will denote by Proj-A the additive category of projective objects in A. And finally, if A, B are additive categories, if T is an autoequivalence of A, and if $\mathsf{For}\colon \mathsf{A} \to \mathsf{B}$ is a functor endowed with an isomorphism $\varepsilon\colon \mathsf{For} \circ T \xrightarrow{\sim} \mathsf{For}$, we will say that For *realizes* A *as a graded version of* B if For is essentially surjective and, for any M, N in A, the natural morphism

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}\big(M, T^n(N)\big) \to \mathrm{Hom}(\mathsf{For} M, \mathsf{For} N) \tag{1.1}$$

induced by For and ε is an isomorphism.

With these notations, some of the main properties of our categories can be roughly stated as follows.

1. ("Beĭlinson-Bernstein localization") There exists an equivalence of abelian categories $\mathsf{P}_{(B)}(\mathscr{B}, \mathbb{C}) \cong \mathscr{O}_0(G)$, where $\mathscr{O}_0(G)$ is the principal block of the category $\mathscr{O}$ of the Lie algebra of G.
2. ("Soergel theory") There exists a functor $\nu\colon \mathsf{IC}_{(\check{B})}(\check{\mathscr{B}}, \mathbb{C}) \to \mathsf{Proj}\text{-}\mathscr{O}_0(G)$ which realizes $\mathsf{IC}_{(\check{B})}(\check{\mathscr{B}}, \mathbb{C})$ (endowed with the shift autoequivalence [1]) as a graded version of $\mathsf{Proj}\text{-}\mathscr{O}_0(G)$.
3. ("Kazhdan-Lusztig conjecture") The multiplicities $[\nabla_w : \mathscr{IC}_v]$ are determined by the specialization at $q = 1$ of a canonical basis of the Hecke algebra $\mathscr{H}_W$ of W.

4. ("Koszul duality") There exists a triangulated category D^{mix} endowed with an autoequivalence and a diagram

$$D^{\mathrm{b}}_{(B)}(\mathscr{B},\mathbb{C}) \longleftarrow D^{\mathrm{mix}} \longrightarrow D^{\mathrm{b}}_{(\check{B})}(\check{\mathscr{B}},\mathbb{C})$$

where both functors are such that (1.1) is an isomorphism for all M, N (for a suitable T), and where simple perverse sheaves on the left correspond to tilting perverse sheaves on the right. As a consequence, the category $\mathsf{P}_{(B)}(\mathscr{B},\mathbb{C})$ is equivalent to the category of (ungraded) modules over a Koszul ring.
5. ("Koszul self-duality") The diagram in (4) is symmetric in the sense that tilting perverse sheaves on the left also correspond to simple perverse sheaves on the right.
6. ("Ringel duality") There exists an autoequivalence of the triangulated category $D^{\mathrm{b}}_{(B)}(\mathscr{B},\mathbb{C})$ sending ∇_w to Δ_{ww_0} and $\mathscr{T}_w$ to $\mathscr{P}_{ww_0}$. (Here, w_0 is the longest element in W.) As a consequence, we have

$$(\mathscr{T}_w : \nabla_v) = (\mathscr{P}_{ww_0} : \Delta_{ww_0}).$$

7. ("formality") If we set $\mathscr{IC}_{\mathscr{B}} := \bigoplus_{w\in W} \mathscr{IC}_w$ and if we consider the graded algebra $E = \big(\bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^n_{D^{\mathrm{b}}_{(B)}(\mathscr{B},\mathbb{C})}(\mathscr{IC}_{\mathscr{B}}, \mathscr{IC}_{\mathscr{B}}[n])\big)^{\mathrm{op}}$ as a differential graded algebra with trivial differential, then there exists an equivalence of triangulated categories

$$D^{\mathrm{b}}_{(B)}(\mathscr{B},\mathbb{C}) \cong E\text{-}\mathsf{dgDerf}$$

where the right-hand side is the derived category of finitely generated differential graded E-modules.

The goal of this series of papers is to give analogues of these properties in the case where $\Bbbk$ is of characteristic $\ell > 0$.

1.4. Known results

First, let us recall what is known about the properties of § 1.3 when $\mathbb{C}$ is replaced by a finite field $\Bbbk$ of characteristic $\ell > 0$. (This case will be referred to as the "modular case," as opposed to the "ordinary case" when $\ell = 0$.)

Property (6) can be immediately generalized, with the same proof (see § 2.3). Property (2) was generalized by Soergel in [28]. Here the main difference with the ordinary case appears: in the modular case the category $\mathsf{IC}_{(\check{B})}(\check{\mathscr{B}},\Bbbk)$ is not well behaved, and the "nice" additive category which should replace $\mathsf{IC}_{(\check{B})}(\check{\mathscr{B}},\mathbb{C})$ is the category $\mathsf{Parity}_{(\check{B})}(\check{\mathscr{B}},\Bbbk)$ of *parity complexes* in the sense of [21]. With this replacement, (2) still holds (when ℓ is bigger than the Coxeter number of G) when $\mathcal{O}_0(G)$ is replaced by Soergel's "modular category $\mathcal{O}$," a certain subquotient of the category of rational representations of a reductive algebraic group over $\Bbbk$ which has the same root datum as G.

Property (7) was also generalized to the modular case (again, where simple perverse sheaves are replaced by parity sheaves) in [24], under the assumption that ℓ is at least the number of roots of G plus 2. Using this result, a representation-theoretic analogue of (4) (which can be obtained, in the ordinary case, by combining properties (1) and (4)) was also obtained in [24], under the same assumptions.

In [24] a second, more technical, difference between the modular setting and the ordinary one appears, related to eigenvalues of the Frobenius. In fact, to obtain a "formality"