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SEGRE CLASSES AND HILBERT SCHEMES OF POINTS

BY ALINA MARIAN, DRAGOS OPREA
AND RAHUL PANDHARIPANDE

ABSTRACT. – We prove a closed formula for the integrals of the top Segre classes of tautological bundles over the Hilbert schemes of points of a $K3$ surface X. We derive relations among the Segre classes via equivariant localization of the virtual fundamental classes of Quot schemes on X. The resulting recursions are then solved explicitly. The formula proves the K-trivial case of a conjecture of M. Lehn from 1999.

The relations determining the Segre classes fit into a much wider theory. By localizing the virtual classes of certain relative Quot schemes on surfaces, we obtain new systems of relations among tautological classes on moduli spaces of surfaces and their relative Hilbert schemes of points. For the moduli of $K3$ surfaces, we produce relations intertwining the κ classes and the Noether-Lefschetz loci. Conjectures are proposed.

RÉSUMÉ. – Nous démontrons une formule explicite pour les intégrales des classes de Segre des fibrés tautologiques sur les schémas de Hilbert ponctuels d'une surface $K3$. À cette fin, on utilise la formule de localisation de la classe virtuelle fondamentale pour certains schémas Quot de Grothendieck associés à la surface $K3$. On en déduit sur les schémas de Hilbert ponctuels des formules récursives entraînant les classes de Segre, qu'on peut résoudre explicitement. Ce résultat établit une conjecture de M. Lehn de 1999, dans le cas des surfaces de classe canonique triviale.

Les récursions décrites ici appartiennent à un tableau beaucoup plus vaste. En localisant les classes virtuelles des schémas Quot relatifs, on obtient des nouvelles relations entre les classes tautologiques dans l'anneau de Chow des espaces des modules des surfaces et de leurs schémas de Hilbert ponctuels relatifs. Pour l'espace des modules des surfaces $K3$, cette méthode donne des relations entre les classes kappa et les classes de Noether-Lefschetz. Nous proposons de nouvelles conjectures sur les anneaux de Chow tautologiques.

0. Introduction

0.1. Segre classes

Let (S, H) be a pair consisting of a nonsingular projective surface S and a line bundle

$$H \to S.$$

 doi:

The *degree* of the pair (S, H) is defined via the intersection product on S,

$$H \cdot H = \int_S H^2 \in \mathbb{Z}.$$

The Hilbert scheme of points $S^{[n]}$ carries a tautological rank n vector bundle $H^{[n]}$ whose fiber over $\zeta \in S^{[n]}$ is given by

$$\zeta \mapsto H^0(H \otimes \mathcal{O}_\zeta).$$

The top Segre class

$$N_{S,H,n} = \int_{S^{[n]}} s_{2n}(H^{[n]})$$

appeared first in the algebraic study of Donaldson invariants via the moduli space of rank 2 bundles on S [25]. Such Segre classes play a basic role in the Donaldson-Thomas counting of sheaves (often entering via the obstruction theory). A classical interpretation of $N_{S,H,n}$ is also available. If $|H|$ is a linear system of dimension $3n-2$ which induces a map

$$S \to \mathbb{P}^{3n-2},$$

$N_{S,H,n}$ counts the n-chords of dimension $n-2$ to the image of S.

The main result of the paper is the calculation of the top Segre classes for all pairs (X, H) in the $K3$ case.

THEOREM 1. – *If (X, H) is a nonsingular $K3$ surface of degree 2ℓ, then*

$$\int_{X^{[n]}} s_{2n}(H^{[n]}) = 2^n \binom{\ell - 2n + 2}{n}.$$

0.2. Lehn's conjecture

Let S be a nonsingular projective surface. The Segre class $N_{S,H,n}$ can be expressed as a polynomial of degree n in the four variables

$$H^2,\ H \cdot K_S,\ K_S^2,\ c_2(S),$$

see [24] for a proof. Furthermore, the form

$$(1) \quad \sum_{n=0}^{\infty} N_{S,H,n}\, z^n = \exp\Big(H^2 \cdot A_1(z) + (H \cdot K_S) \cdot A_2(z) + K_S^2 \cdot A_3(z) + c_2(S) \cdot A_4(z)\Big)$$

in terms of four universal power series A_1, A_2, A_3, A_4 was proven in [7]. The formulas for the four power series were explicitly conjectured by M. Lehn in 1999.

CONJECTURE 1 (Lehn [14]). – *We have*

$$(2) \qquad \sum_{n=0}^{\infty} N_{S,H,n}\, z^n = \frac{(1-w)^a (1-2w)^b}{(1-6w+6w^2)^c}$$

for the change of variable

$$z = \frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3}$$

and constants

$$a = H \cdot K_S - 2K_S^2,\quad b = (H - K_S)^2 + 3\chi(\mathcal{O}_S),\quad c = \frac{1}{2}H(H - K_S) + \chi(\mathcal{O}_S).$$

As usual in the study of the Hilbert scheme of points, Theorem 1 determines two of the power series in (1). Specifically, Theorem 1 implies

$$A_1\left(\frac{1}{2}t(1+t)^2\right) = \frac{1}{2}\log(1+t), \tag{3}$$

$$A_4\left(\frac{1}{2}t(1+t)^2\right) = \frac{1}{8}\log(1+t) - \frac{1}{24}\log(1+3t). \tag{4}$$

The evaluation of A_1 and A_4 proves Lehn's conjecture for all surfaces with numerically trivial canonical bundle.

Corollary 1. – *If (A, H) is an abelian or bielliptic surface of degree 2ℓ, then*

$$\int_{A^{[n]}} s_{2n}(H^{[n]}) = \frac{2^n \ell}{n}\binom{\ell - 2n - 1}{n - 1}.$$

Corollary 2. – *If (E, H) is an Enriques surface of degree 2ℓ, then*

$$\left(\sum_{n=0}^{\infty} N_{E,H,n}\, z^n\right)^2 = \sum_{n=0}^{\infty} 2^n \binom{2\ell - 2n + 2}{n} z^n.$$

0.3. Strategy of the proof

The intersection theory of the Hilbert scheme of points can be approached via the inductive recursions set up in [7] or via the Nakajima calculus [14, 19]. By these methods, the integration of tautological classes is reduced to a combinatorial problem. Another strategy is to prove an equivariant version of Lehn's conjecture for the Hilbert scheme of points of $\mathbb{C}^2$ via appropriately weighted sums over partitions. However, we do not know how to prove Theorem 1 along these lines. (1)

Let (X, H) be a nonsingular projective $K3$ surface. We consider integrals over the Quot scheme $\mathcal{Q}_{H,\chi}(\mathbb{C}^2)$ parametrizing quotients

$$\mathbb{C}^2 \otimes \mathcal{O}_X \to F \to 0$$

where F is a rank 0 coherent sheaf satisfying

$$c_1(F) = H \quad \text{and} \quad \chi(F) = \chi.$$

The Quot scheme admits a reduced virtual class, and the integrals

$$\int_{[\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\mathrm{red}}} \gamma \cdot 0^k$$

vanish for all $k > 0$ and for all choices of Chow classes γ. Here, the notation 0 stands for the first Chern class of the trivial line bundle

$$c_1(\mathcal{O}) = 0 \in A^1(\mathcal{Q}_{H,\chi}(\mathbb{C}^2)).$$

(1) The parallel problem in dimension 1, the calculation of the Segre classes of tautological bundles over Hilbert schemes of points of nonsingular *curves*, has been solved in [5, 13, 26].