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WELL-POSEDNESS AND DISPERSIVE DECAY OF SMALL DATA SOLUTIONS FOR THE BENJAMIN-ONO EQUATION

BY MIHAELA IFRIM AND DANIEL TATARU

ABSTRACT. – This article represents a first step toward understanding the long time dynamics of solutions for the Benjamin-Ono equation. While this problem is known to be both completely integrable and globally well-posed in L^2, much less seems to be known concerning its long time dynamics. Here, we prove that for small localized data the solutions have (nearly) dispersive dynamics almost globally in time. An additional objective is to revisit the L^2 theory for the Benjamin-Ono equation and provide a simpler, self-contained approach.

RÉSUMÉ. – Cet article représente une première étape vers la compréhension du comportement en temps long pour l'équation de Benjamin-Ono. Tandis que ce problème est à la fois complètement intégrable et globalement bien posé en L^2, beaucoup moins semble être connu en ce qui concerne son comportement en temps long. Nous montrons ici que pour de données petites et localisées, les solutions ont une dynamique dispersive presque globalement en temps. Un autre objectif est de revoir la théorie L^2 pour Benjamin-Ono et de fournir une approche plus simple et autonome.

1. Introduction

In this article we consider the Benjamin-Ono equation

$$(\partial_t + H\partial_x^2)\phi = \frac{1}{2}\partial_x(\phi^2), \qquad \phi(0) = \phi_0, \tag{1.1}$$

where ϕ is a real valued function $\phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. H denotes the Hilbert transform on the real line; we use the convention that its symbol is

$$H(\xi) = -i \operatorname{sgn} \xi$$

as in Tao [37] and opposite to Kenig-Martel [25]. Thus, dispersive waves travel to the right and solitons to the left.

The Benjamin-Ono equation is a model for the propagation of one dimensional internal waves (see [4]). Among others, it describes the physical phenomena of wave propagation at the interface of layers of fluids with different densities (see Benjamin [4] and Ono [30]). It also

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belongs to a larger class of equations modeling this type of phenomena, some of which are certainly more physically relevant than others.

Equation (1.1) is known to be completely integrable. In particular it has an associated Lax pair, an inverse scattering transform, and an infinite hierarchy of conservation laws. For further information in this direction we refer the reader to [23] and references therein. We list only some of these energies, which are easily verified to be conserved for regular solutions (for example $H^3_x(\mathbf{R})$). Integrating by parts, one sees that this problem has conserved mass,

$$E_0 = \int \phi^2 \, dx,$$

momentum

$$E_1 = \int \left(\phi H \phi_x - \frac{1}{3}\phi^3 \right) dx,$$

as well as energy

$$E_2 = \int \left(\phi_x^2 - \frac{3}{4}\phi^2 H \phi_x + \frac{1}{8}\phi^4 \right) dx.$$

More generally, at each nonnegative integer k we similarly have a conserved energy E_k corresponding at leading order to the $\dot{H}^{\frac{k}{2}}$ norm of ϕ.

This is closely related to the Hamiltonian structure of the equation, which uses the symplectic form

$$\omega(\psi_1, \psi_2) = \int \left(\psi_1 \partial_x^{-1} \psi_2 - \psi_2 \partial_x^{-1} \psi_1 \right) dx$$

with associated map $J = \partial_x$. Then the Benjamin-Ono equation is generated by the Hamiltonian E_1 and symplectic form ω. E_0 generates the group of translations. All higher order conserved energies can be viewed in turn as Hamiltonians for a family of commuting flows, which are known as the Benjamin-Ono hierarchy of equations.

The Benjamin-Ono equation is a dispersive equation, i.e., the group velocity of waves depends on the frequency. Precisely, the dispersion relation for the linear part is given by

$$\omega(\xi) = -\xi|\xi|,$$

and the group velocity for waves of frequency ξ is $v = 2|\xi|$. Here we are considering real solutions, so the positive and negative frequencies are matched. However, if one were to restrict the linear Benjamin-Ono waves to either positive or negative frequencies then we obtain a linear Schrödinger equation with a choice of signs. Thus one expects that many features arising in the study of nonlinear Schrödinger equations will also appear in the study of the Benjamin-Ono equation.

Last but not least, when working with the Benjamin-Ono equation one has to take into account its quasilinear character. A cursory examination of the equation might lead one to the conclusion that it is in effect semilinear. It is only a deeper analysis (see [29], [27]) which reveals the fact that the derivative in the nonlinearity is strong enough to insure that the nonlinearity is non-perturbative, and that only continuous dependence on the initial data may hold, even at high regularity.

Considering local and global well-posedness results in Sobolev spaces H^s, a natural threshold is given by the fact that the Benjamin-Ono equation has a scale invariance,

$$\phi(t,x) \to \lambda\phi(\lambda^2 t, \lambda x), \tag{1.2}$$

and the scale invariant Sobolev space in dimension 1 associated to this scaling is $\dot{H}^{-\frac{1}{2}}$.

There have been many developments in the well-posedness theory for the Benjamin-Ono equations, see: [6, 21, 27, 24, 37, 29, 32, 22, 33]. Well-posedness in weighted Sobolev spaces was considered in [9] and [8], while soliton stability was studied in [25, 10]. These is also closely related work on an extended class of equations, called the generalized Benjamin-Ono equations, for which we refer the reader to [12], [13] and references therein. A more extensive discussion of the Benjamin-Ono equation and related fluid models can be found in the survey papers [1] and [26].

Presently, for the Cauchy problem at low regularity, the existence and uniqueness result at the level of $H^s(\mathbf{R})$ data is known for the Sobolev index $s \geq 0$. Well-posedness in the range $-\frac{1}{2} \leq s < 0$ appears to be an open question. We now review some of the key thresholds in this analysis.

The H^3 well-posedness result was obtained by Saut in [33], using energy estimates. For convenience we use his result as a starting point for our work, which is why we recall it here:

THEOREM 1. – *The Benjamin-Ono equation is globally well-posed in H^3.*

The H^1 threshold is another important one, and it was reached by Tao [37]; his article is highly relevant to the present work, and it is where the idea of renormalization is first used in the study of the Benjamin-Ono equation.

The L^2 threshold was first reached by Ionescu and Kenig [21], essentially by implementing Tao's renormalization argument in the context of a much more involved and more delicate functional setting, inspired in part from the work of the second author [38] and of Tao [37] on wave maps. This is imposed by the fact that the derivative in the nonlinearity is borderline from the perspective of bilinear estimates, i.e., there is no room for high frequency losses. An attempt to simplify the L^2 theory was later made by Molinet-Pilod [28]; however, their approach still involves a rather complicated functional structure, using not only $X^{s,b}$ spaces but additional weighted mixed norms in frequency.

Our first goal here is to revisit the L^2 theory for the Benjamin-Ono equation, and (re)prove the following theorem:

THEOREM 2. – *The Benjamin-Ono equation is globally well-posed in L^2.*

Since the L^2 norm of the solutions is conserved, this is in effect a local in time result, trivially propagated in time by the conservation of mass. In particular it says little about the long time properties of the flow, which will be our primary target here.

The proof we give here is for the case of the Benjamin-Ono equation on the real line. However, it can be easily adapted to the periodic setting.

Given the quasilinear nature of the Benjamin-Ono equation, here it is important to specify the meaning of well-posedness. This is summarized in the following properties:

(i) *Existence of regular solutions*: For each initial data $\phi_0 \in H^3$ there exists a unique global solution $\phi \in C(\mathbf{R}; H^3)$.