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STRUCTURE AT INFINITY FOR SHRINKING RICCI SOLITONS

BY Ovidiu MUNTEANU AND Jiaping WANG

Abstract. – This paper concerns the structure at infinity for complete gradient shrinking Ricci solitons. It is shown that for such a soliton with bounded curvature, if the round cylinder $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$ occurs as a limit for a sequence of points going to infinity along an end, then the end is asymptotic at infinity to the same round cylinder. This result is applied to obtain structural results at infinity for four dimensional gradient shrinking Ricci solitons. It was previously known that such solitons with scalar curvature approaching zero at infinity must be smoothly asymptotic to a cone. For the case that the scalar curvature is bounded from below by a positive constant, we conclude that along each end the soliton is asymptotic to a quotient of $\mathbb{R} \times \mathbb{S}^3$ or converges to a quotient of $\mathbb{R}^2 \times \mathbb{S}^2$ along each integral curve of the gradient vector field of the potential function.

Résumé. – Cet article concerne principalement la structure à l'infini pour gradient solitons rétrécis de Ricci. Il est montré que pour un tel soliton avec courbure bornée, si le cylindre rond $\mathbb{R} \times \mathbb{S}^{n-1}/\Gamma$ se produit comme une limite pour une séquence de points convergeant à l'infini le long d'une extrémité, alors l'extrémité est asymptotique au même cylindre rond à l'infini. Le résultat est appliqué pour obtenir des résultats structurels à l'infini pour gradient solitons de Ricci de dimension quatre. On sait déjà que ces solitons avec courbure scalaire proche de zéro à l'infini doivent être asymptotiques à un cône. Dans le cas où la courbure scalaire est délimitée par le bas par une constante positive, nous concluons que le long de chaque extrémité le soliton est asymptotique à un quotient de $\mathbb{R} \times \mathbb{S}^3$ ou converge vers un quotient de $\mathbb{R}^2 \times \mathbb{S}^2$ le long de chaque courbe intégrale du champ de vecteur de gradient de la fonction potentielle.

1. Introduction

The goal of this paper is to continue our study of complete four dimensional gradient shrinking Ricci solitons initiated in [25] and to obtain further information concerning the structure at infinity of such manifolds. Recall that a Riemannian manifold (M, g) is a

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gradient shrinking Ricci soliton if there exists a smooth function $f \in C^{\infty}(M)$ such that the Ricci curvature Ric of M and the hessian Hess(f) of f satisfy the following equation

$$\mathrm{Ric} + \mathrm{Hess}\,(f) = \frac{1}{2} g.$$

By defining ϕ_t to be the one-parameter family of diffeomorphisms generated by the vector field $\frac{\nabla f}{-t}$ for $-\infty < t < 0$, one checks that $g(t) = (-t)\,\phi_t^* \, g$ is a solution to the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(t)$$

on time interval $(-\infty, 0)$. Since the Ricci flow equation is invariant under the action of the diffeomorphism group, such solution $g(t)$ is evidently a shrinking self-similar solution to the Ricci flow. Gradient shrinking Ricci solitons have played a crucial role in the singularity analysis of Ricci flows. A conjecture, generally attributed to Hamilton, asserts that the blow-ups around a type-I singularity point of a Ricci flow always converge to (nontrivial) gradient shrinking Ricci solitons. More precisely, a Ricci flow solution $(M, g(t))$ on a finite-time interval $[0, T)$, $T < \infty$, is said to develop a Type-I singularity (and T is called a Type-I singular time) if there exists a constant $C > 0$ such that for all $t \in [0, T)$

$$\sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} \leq \frac{C}{T-t}$$

and

$$\limsup_{t\to T} \; \sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} = \infty.$$

Here $\mathrm{Rm}_{g(t)}$ denotes the Riemannian curvature tensor of the metric $g(t)$. A point $p \in M$ is a singular point if there exists no neighborhood of p on which $|\mathrm{Rm}_{g(t)}|_{g(t)}$ stays bounded as $t \to T$. Then the conjecture claims that for every sequence $\lambda_j \to \infty$, the rescaled Ricci flows $(M, g_j(t), p)$ defined on $[-\lambda_j T, 0)$ by $g_j(t) := \lambda_j \, g(T + \lambda_j^{-1} t)$ subconverge to a nontrivial gradient shrinking Ricci soliton.

While the conjecture was first confirmed by Perelman [29] for the dimension three case, in the most general form it has also been satisfactorily resolved. In the case where the blow-up limit is compact, it was confirmed by Sesum [32]. In the general case, blow-up to a gradient shrinking soliton was proved by Naber [27]. The nontriviality issue of the soliton was later taken up by Enders, Müller and Topping [15], see also Cao and Zhang [8].

In view of their importance, it is then natural to seek a classification of the gradient shrinking Ricci solitons. It is relatively simple to classify two dimensional ones, [18].

THEOREM 1.1. – *A two dimensional gradient shrinking Ricci soliton is isometric to the plane* $\mathbb{R}^2$ *or to a quotient of the sphere* $\mathbb{S}^2$.

For the three dimensional case, there is a parallel classification result as well.

THEOREM 1.2. – *A three dimensional gradient shrinking Ricci soliton is isometric to the Euclidean space* $\mathbb{R}^3$ *or to a quotient of the sphere* $\mathbb{S}^3$ *or of the cylinder* $\mathbb{R} \times \mathbb{S}^2$.

This theorem has a long history. Ivey [21] first showed that a three dimensional compact gradient shrinking Ricci soliton must be a quotient of the sphere $\mathbb{S}^3$. Later, it was realized from the Hamilton-Ivey estimate [18] that the curvature of a three dimensional gradient shrinking Ricci soliton must be nonnegative. Moreover, by the strong maximum principle of Hamilton [17], the manifold must split off a line, hence is a quotient of $\mathbb{R} \times \mathbb{S}^2$ or $\mathbb{R}^3$, if its sectional curvature is not strictly positive. When the sectional curvature is strictly positive, Perelman [30] showed that the soliton must be compact, hence a quotient of the sphere, provided that the soliton is noncollapsing with bounded curvature. Obviously, the classification result follows by combining all these together, at least for the ones which are noncollapsing with bounded curvature. The result in particular implies that a type I singularity of the Ricci flow on a compact three dimensional manifold is necessarily spherical or neck-like, a fact crucial for Perelman [30] to define the Ricci flows with surgery and for the eventual resolution of the Poincaré or the more general Thurston's geometrization conjecture. The noncollapsing assumption was later removed by Naber [27]. By adopting a different argument, Ni and Wallach [28], and Cao, Chen and Zhu [3] showed the full classification result theorem 1.2. Some relevant contributions were also made in [27, 31]. In passing, we mention that it is now known that a complete shrinking Ricci soliton of any dimension with positive sectional curvature is compact by [26].

The logical next step is to search for a classification of four dimensional gradient shrinking Ricci solitons. Such a result should be very much relevant in understanding the formation of singularities of the Ricci flows on four dimensional manifolds, just like the three dimensional case. However, in contrast to the dimension three case, for dimension four or higher, the curvature of a gradient shrinking Ricci soliton may change sign as demonstrated by the examples constructed in [16]. The existence of such examples, which are obviously not of the form of a sphere, or the Euclidean space, or their product, certainly complicates the classification outlook.

Note that in the case of dimension three, the curvature operator, being nonnegative, is bounded by the scalar curvature. In the case of dimension four, we showed that such a conclusion still holds even though the curvature operator no longer has a fixed sign, [25]. In particular, this implies that the curvature operator must be bounded if the scalar curvature is.

THEOREM 1.3. – *Let (M, g, f) be a four dimensional complete gradient shrinking Ricci soliton with bounded scalar curvature S. Then there exists a constant $c > 0$ so that*

$$|\mathrm{Rm}| \leq c\, S \quad \text{on } M.$$

In the theorem, the constant $c > 0$ depends only on the upper bound of the scalar curvature A and the geometry of the geodesic ball $B_p(r_0)$, where p is a minimum point of potential function f and r_0 is determined by A. We stress that the potential function f of the soliton is exploited in an essential way in our proof by working on the level sets of f.

As an application, we obtained the following structural result. Recall that a Riemannian cone is a manifold $[0, \infty) \times \Sigma$ endowed with Riemannian metric $g_c = dr^2 + r^2 g_\Sigma$, where (Σ, g_Σ) is a closed $(n-1)$-dimensional Riemannian manifold. Denote $E_R = (R, \infty) \times \Sigma$ for $R \geq 0$ and define the dilation by λ to be the map $\rho_\lambda : E_0 \to E_0$ given by $\rho_\lambda(r, \sigma) =$