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LOWER BOUNDS FOR RANKS OF MUMFORD-TATE GROUPS

BY MARTIN ORR

ABSTRACT. — Let A be a complex abelian variety and G its Mumford-Tate group. Supposing that the simple abelian subvarieties of A are pairwise non-isogenous, we find a lower bound for the rank $\operatorname{rk} G$ of G, which is a little less than $\log_2 \dim A$. If we suppose furthermore that $\operatorname{End} A$ is commutative, then we can improve this lower bound to $\operatorname{rk} G \geq \log_2 \dim A + 2$ and prove that this is sharp. We also obtain the same results for the rank of the ℓ-adic monodromy group of an abelian variety defined over a number field.

RÉSUMÉ (*Minoration des rangs de groupes de Mumford-Tate*). — Soit A une variété abélienne complexe et G son groupe de Mumford-Tate. En supposant que les sous variétés abéliennes simples de A sont deux à deux non-isogènes, on trouve une minoration du rang $\operatorname{rk} G$ de G, légèrement inférieure à $\log_2 \dim A$. Si de plus on suppose que $\operatorname{End} A$ est commutatif, alors on peut améliorer cette borne en $\operatorname{rk} G \geq \log_2 \dim A + 2$, et montrer que cette borne-ci est optimale. On obtient les mêmes resultats pour le rang du groupe de monodromie ℓ-adique d'une variété abélienne définie sur un corps de nombres.

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1. Introduction

Let A be a complex abelian variety of dimension g, whose simple abelian subvarieties are pairwise non-isogenous. In this paper we will establish a lower bound for the rank of the Mumford-Tate group of A. The Mumford-Tate group is an algebraic group over $\mathbb{Q}$ defined via the Hodge theory of A (see Section 2 below for the definition). The same argument will also establish a lower bound for the rank of the ℓ-adic monodromy groups G_ℓ, in the case where A is defined over a number field. The ℓ-adic monodromy group is the Zariski closure of the image of the Galois representation on the ℓ-adic Tate module of A. Our main theorems are the following:

THEOREM 1.1. — *Let A be an abelian variety of dimension g such that* $\operatorname{End} A$ *is commutative. Let G be the Mumford-Tate group or the ℓ-adic monodromy group of A. Then* $\operatorname{rk} G \geq \log_2 g + 2$.

THEOREM 1.2. — *Let A be an abelian variety of dimension g whose simple abelian subvarieties are pairwise non-isogenous. Let G be the Mumford-Tate group or the ℓ-adic monodromy group of A. If $n = \operatorname{rk} G$, then*

$$n + \alpha(n)\sqrt{n \log n} \geq \log_2 g + 2$$

for a function $\alpha : \mathbb{N}_{\geq 2} \to \mathbb{R}$ satisfying $\alpha(n) < 2$ for all n and $\alpha(n) \to 1/\log 2 = 1.44...$ as $n \to \infty$.

Each of these theorems is an instance of a more general bound for weak Mumford-Tate triples, which are defined in Section 2. These more general bounds are Theorems 4.1 and 4.4 respectively. These would apply also for example to the analogue of the Mumford-Tate group for a Hodge-Tate module of weights 0 and 1.

Theorem 1.1 was proved by Ribet in the case of an abelian variety with complex multiplication [13]. Our proof is a generalisation of his, relying on the fact that the defining representation of the Mumford-Tate group or ℓ-adic monodromy group has minuscule weights.

The condition on simple subvarieties in Theorem 1.2 is necessary: taking products of copies of the same simple abelian variety increases the dimension without changing the rank of the Mumford-Tate group. Indeed, if A is isogenous to $\prod_i A_i^{m_i}$ where the A_i are simple and pairwise non-isogenous, then according to [4] Lemme 2.2,

$$\mathrm{MT}(A) \cong \mathrm{MT}(\prod_i A_i).$$

Hence Theorem 1.2 implies that for a general abelian variety A, if n denotes the rank of either the Mumford-Tate group or the ℓ-adic monodromy group of A, then

$$n + \alpha(n)\sqrt{n \log n} \geq \log_2 \left(\sum_i \dim A_i \right) + 2$$

where the A_i are one representative of each isogeny class of simple abelian subvarieties of A.

The condition of having pairwise non-isogenous simple abelian subvarieties can be interpreted via the endomorphism algebra like the condition in Theorem 1.1: it is equivalent to $\operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$ being a product of division algebras. Note also that $\operatorname{End} A$ being commutative implies the condition of Theorem 1.2. (Throughout this paper, $\operatorname{End} A$ means the endomorphisms of A after extension of scalars to an algebraically closed field.)

Let G be either the Mumford-Tate group or the ℓ-adic monodromy group of A. It is well known that the rank of G is at most $g + 1$, and that this upper bound is achieved for a generic abelian variety. Indeed, if g is odd and $\operatorname{End} A = \mathbb{Z}$, then $\operatorname{rk} G$ is always $g + 1$ [16]. So in this case the bound in Theorem 1.1 is far from sharp.

On the other hand if g is a power of 2, then there are abelian varieties for which the bound in Theorem 1.1 is achieved (even with $\operatorname{End} A = \mathbb{Z}$). We construct such examples in Section 5. The exact bound for a given g is very sensitive to the prime factors of g. Equality can happen only when g is a power of 2 (for the trivial reason that otherwise $\log_2 g \notin \mathbb{Z}$) but even near-equality can only occur when g has many small prime factors. This was made precise by Dodson in the complex multiplication case [2], and it is possible that something similar could be proved in general.

Theorem 1.2 is not sharp. The function $\alpha(n)$ is specified exactly in Section 4, but it is likely that this could be improved on, perhaps to something which goes to 0 as $n \to \infty$. In Section 5, we construct a family of examples showing that Theorem 1.2 cannot be improved to $n + k \geq \log_2 g$ for any constant k.

We can deduce a lower bound for the growth of the degrees of the division fields $K(A[\ell^n])$ (for ℓ a fixed prime number) as a straightforward consequence of Theorem 1.1.

COROLLARY 1.3. — *Let A be an abelian variety of dimension g over a number field K, and ℓ a prime number. If* $\operatorname{End} A$ *is commutative, then there is a constant $C(A, K, \ell)$ such that*

$$[K(A[\ell^n]) : K] \geq C(A, K, \ell) \cdot \ell^{n(\log_2 g + 2)}.$$