MÉMOIRES DE LA SMF 100

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Société Mathématique de France 2005
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification*. — Primary 11P05, 11P55; Secondary 11D45, 11D85, 11L15, 11N56.

***Key words and phrases*. —** Waring's problem, circle method, Weyl sums, multiplicative functions, Diophantine equations.

J.-M. D.: Supported by the universities Bordeaux 1, Victor Segalen Bordeaux 2 and CNRS (UMR 5465).
T.D. W.:Packard Fellow, and supported in part by NSF grant DMS-9970440.

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Abstract. **—** By 1939 it was known that 13792 cannot be expressed as a sum of sixteen biquadrates (folklore), that there exist infinitely many natural numbers which cannot be written as sums of fifteen biquadrates (Kempner) and that every sufficiently large integer is a sum of sixteen biquadrates (Davenport).
In this memoir it is shown that every integer larger than 10^{216} and not divisible by 16 can be represented as a sum of sixteen biquadrates. Combined with a numerical study by Deshouillers, Hennecart and Landreau, this result implies that every integer larger than 13792 is a sum of sixteen biquadrates.

Résumé **(Sur les Sommes de Seize Bicarrés). —** En 1939, on savait que 13792 ne peut pas être représenté comme somme de seize bicarrés (folklore), qu'il existe une infinité d'entiers qui ne peuvent pas être écrits comme sommes de quinze bicarrés (Kempner) et que tout entier assez grand est somme de seize bicarrés (Davenport).
Dans ce mémoire, on montre que tout entier supérieur à 10^{216} et non divisible par 16 peut s'exprimer comme somme de seize bicarrés. Combiné à une étude numérique menée par Deshouillers, Hennecart et Landreau, ce résultat implique que tout entier supérieur à 13792 est somme de seize bicarrés.

CONTENTS

CHAPTER 1

INTRODUCTION

The investigation of sums of biquadrates occupies a distinguished position in additive number theory, largely on account of the relative success with which the basic problems of Waring-type have been addressed. Although progress on such problems was dominated for the greater part of the 20th century by advances in technology at the heart of the Hardy-Littlewood method, the older ideas involving the use of polynomial identities have recently resurfaced in work of Kawada and Wooley [**15**], though now interwoven with the analytic machinery of the circle method itself. The primary goal of this paper is to apply this new circle of ideas to obtain an explicit analysis of sums of sixteen biquadrates, and, moreover, one suitable for determining the largest integer not represented in such a manner. A separate paper [**12**] reports on computations of Deshouillers, Hennecart and Landreau which complement the main conclusion of this memoir, and as will shortly become apparent, the union of these results leads to the following definitive statement concerning sums of sixteen biquadrates.

Theorem 1. — *Every integer exceeding* 13792 *can be written as a sum of at most 16 biquadrates.*

Although we avoid a detailed historical account of the various contributions to Waring's problem for biquadrates, our subsequent discussion will be clarified by a sketchy overview of such matters (we refer the reader to the survey [**11**] for a more comprehensive account). For the sake of concision, we refer to a number n as being a B_s (number) when n can be written as a sum of at most s biquadrates. In accordance with the familiar notation in Waring's problem, we then denote by $g(4)$ the least integer s with the property that every natural number is a B_s, and we denote by $G(4)$ the least natural number s such that every sufficiently large number is a B_s. The problem central to this paper has as its origin the assertion made by Waring in 1770 to the effect that $g(4) = 19$. This conjecture was in large part resolved by Hardy and

Littlewood [**13**], who established by means of their newly devised circle method that $G(4) \leqslant 19$. Indeed, the work of Hardy and Littlewood shows that one may compute an explicit constant C with the property that every number exceeding C is a B_{19}. Although a computational check of the integers of size at most C would determine whether or not $g(4)$ is equal to 19, the astronomical size of this constant C entirely precluded any such attempt to resolve this problem. While for other exponents k, advances in the circle method rapidly wrought an effective determination of the value of $g(k)$, it was only in the late 1980's that, with new ideas and substantial effort, it became possible to reduce the value of C to a size within the grasp of existing supercomputers. Thus Balasubramanian, Deshouillers and Dress at last announced a proof of $g(4) = 19$ in [**3**], [**4**]. A complete proof of the result can be found in the series of papers [**7**], [**8**], [**9**] and [**10**].

While it has only recently been established that every natural number is a B_{19}, as Waring had claimed, it has been known for many years that $G(4)$ is less than 19. Indeed, Davenport [**5**] had shown by 1939 that $G(4) = 16$, so that with only finitely many exceptions, all natural numbers are B_{16}. We recall at this point that the lower bound $G(4) \geqslant 16$ is immediate from the observation that $31 \cdot 16^m$ is not a B_{15} for any non-negative integer m. As announced in [**11**], by combining the work of Balasubramanian, Deshouillers and Dress with the central idea of the recent memoir [**15**] of Kawada and Wooley, it is now possible to determine all numbers that are not B_{16}. The object of this treatise is the detailed proof of the following result.

THEOREM 2. — *Every integer exceeding* 10^{216} *that is not divisible by 16 can be written as the sum of 16 biquadrates.*

A companion paper of Deshouillers, Hennecart and Landreau [**12**] shows that all natural numbers not exceeding 10^{245} are B_{16}, with the exception of precisely 96 numbers, the largest of which is 13792. In view of the latter conclusion, Theorem 1 follows from Theorem 2 by noting that integers exceeding 10^{216} divisible by 16 are harmless. For if $N > 10^{216}$ and $16|N$, then there exist natural numbers m and n with the property that $N = 16^m n$, and either $n > 10^{216}$ and $16 \nmid n$, or else $10^{216}/16 < n \leqslant 10^{216}$. In the former case, Theorem 2 shows that n is a B_{16}, and in the latter case the above cited conclusion of Deshouillers, Hennecart and Landreau [**12**] shows that n is a B_{16}. Thus, in either case, it is evident that $N = (2^m)^4 n$ is a B_{16}.

We remark that Deshouillers, Hennecart and Landreau [**12**] have determined in addition the 31 numbers that are not B_{17} (the largest of which is 1248), and also the 7 numbers that are not B_{18}, these being simply described as the integers $80k - 1$ for $1 \leqslant k \leqslant 7$. We refer the reader to the aforementioned paper [**12**] for a complete list of the exceptional numbers which are not B_{16}, and those which are not B_{17} (this information may also be found in the survey [**11**]).

We next provide a brief overview of our basic strategy, deferring to section 2 a more detailed discussion of our plan of attack on the proof of Theorem 2. We employ the Hardy–Littlewood method, aiming to exploit the polynomial identity

$$x^4 + y^4 + (x+y)^4 = 2(x^2 + xy + y^2)^2 \tag{1.1}$$

that was the key innovation of Kawada and Wooley [**15**]. In order to efficiently exploit the relation (1.1), we introduce the set $\mathcal{M}$, which we define by

$$\mathcal{M} = \{ m \in \mathbb{N} : \; m = x^2 + xy + y^2 \; \text{ for some } x,\, y \in \mathbb{Z} \text{ with } xy(x+y) \neq 0 \,\}. \tag{1.2}$$

In view of (1.1), for each $m \in \mathcal{M}$ one finds that $2m^2$ is a sum of 3 biquadrates. Thus one is led to consider the number, $\mathcal{Z}(X)$, of solutions of the equation

$$2m_1^2 + u_1^4 + u_2^4 = 2m_2^2 + u_3^4 + u_4^4,$$

with $m_1, m_2 \in \mathcal{M} \cap [1, X^{1/2}]$ and $1 \leqslant u_i \leqslant X^{1/4}$ $(1 \leqslant i \leqslant 4)$. By employing a modified divisor function estimate to determine the number of solutions of the latter equation with $u_1^4 + u_2^4 \neq u_3^4 + u_4^4$, and an immediate counting argument when $u_1^4 + u_2^4 = u_3^4 + u_4^4$, one derives the efficient upper bound $\mathcal{Z}(X) \ll X(\log X)^{\varepsilon}$ without any undue effort (see the proof of Theorem 1 in Kawada and Wooley [**15**, § 2], and also the related discussion of Lemma 3.4 of [**15**]).

In order to establish that a given large number N is a B_{16}, the most obvious strategy suggested by the above discussion is that of considering representations of N in the form

$$N = 2m_1^2 + 2m_2^2 + x_1^4 + \cdots + x_{10}^4, \tag{1.3}$$

with $m_1,\ m_2 \in \mathcal{M}$ and $x_j \in \mathbb{N}$ $(1 \leqslant j \leqslant 10)$. It is now apparent that whenever N admits a representation of the shape (1.3), then N may be written as the sum of 16 biquadrates. Unfortunately, since a biquadrate is congruent to 0 or 1 modulo 16 according to whether it is even or odd, one finds from (1.1) that whenever $m \in \mathcal{M}$, the expression $2m^2$ is necessarily congruent to 0 or 2 modulo 16. Thus, whereas an unrestricted sum of three biquadrates is congruent to 0, 1, 2 or 3 modulo 16, our surrogate $2m^2$ is restricted to the classes 0 and 2 modulo 16. It follows that whether or not the integer N is a B_{16}, it fails to possess a representation in the shape (1.3) whenever $N \equiv 15 \pmod{16}$, and thus our initial strategy is doomed to failure. Nonetheless, by making use of the tools developed within this memoir, the authors have employed this approach to establish that whenever $N \geqslant 10^{156}$, and N is not congruent to 0 or 15 modulo 16, then N can be written in the shape (1.3), and hence is a B_{16}. We omit the details of such an argument in the interest of saving space.

As is apparent from the deliberations of the previous paragraph, one may recover the missing congruence class 15 modulo 16 by considering instead representations of N in the form

$$N = 2m^2 + x_1^4 + \cdots + x_{13}^4, \tag{1.4}$$

with $m \in \mathcal{M}$ and $x_j \in \mathbb{N}$ $(1 \leqslant j \leqslant 13)$. By combining the mean value estimate discussed above with an explicit version of Hua's inequality provided by Deshouillers and Dress [**9**], a careful application of the Hardy-Littlewood method would establish that whenever $N \geqslant 10^{300}$ and $16 \nmid N$, then N is a B_{16}. Unfortunately, even anticipated advances in computational technology would seem insufficient to permit the methods of Deshouillers, Hennecart and Landreau [**12**] to check that all numbers not exceeding 10^{300} are B_{16}, with the above-mentioned exceptions. We are therefore forced in our proof of Theorem 2 to introduce a further new idea.

Motivated by the identity (1.1), and the similar identity

$$x^2 + y^2 + (x+y)^2 = 2(x^2 + xy + y^2),$$

we obtain from the relation

$$(w+x)^4 + (w-x)^4 = 2w^4 + 12w^2x^2 + 2x^4$$

the new identity

$$\begin{aligned}(w+x)^4 + (w-x)^4 + &(w+y)^4 + (w-y)^4 + (w+x+y)^4 + (w-x-y)^4 \\ &= 4(x^2+xy+y^2)^2 + 24(x^2+xy+y^2)w^2 + 6w^4 \\ &= 4(x^2+xy+y^2+3w^2)^2 - 30w^4.\end{aligned} \tag{1.5}$$

The use of (1.1) in the representations (1.3) and (1.4) might reasonably be regarded as effectively replacing three biquadrates by a square. The use of the identity (1.5), meanwhile, effectively replaces six biquadrates by a square and a biquadrate, which in applications amounts to trading five biquadrates for a square. While the latter exchange is clearly less efficient than the former so far as consequent mean value estimates are concerned (see Lemmata 2.4 and 2.5 below), in compensation one finds that the six biquadrates on the left hand side of (1.5) may be simultaneously odd. Moreover, despite the relative inefficiency of the identity (1.5) as compared to (1.1), one may nonetheless recover a mean value estimate associated with only 14 biquadrates of essentially the same strength as that available from Hua's inequality for 16 biquadrates (compare Theorem 4 of Deshouillers and Dress [**9**] with Lemma 2.5 below). Thus it transpires that the new identity (1.5) is crucial to the success of this paper.

In order to establish that a given large integer N is a B_{16}, therefore, the strategy which we adopt in this memoir is to consider representations of N in the form

$$N = 2m_1^2 + 4m_2^2 + 24m_2w^2 + 6w^4 + x_1^4 + \cdots + x_7^4,$$

with $m_1,\ m_2 \in \mathcal{M}$ and $w, x_j \in \mathbb{N}$ $(1 \leqslant j \leqslant 7)$. In view of the identities (1.1) and (1.5), it follows that whenever N can be written in the latter form, then N is necessarily a sum of 16 biquadrates. A discussion of the details associated with putting this strategy into practice may be found in § 2 below, wherein an outline of the proof of Theorem 2 is also provided.