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Ruhan ZHAO and Kehe ZHU

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THEORY OF BERGMAN SPACES IN THE UNIT BALL OF $\mathbb{C}^n$

Ruhan Zhao, Kehe Zhu

Abstract. – There has been a great deal of work done in recent years on weighted Bergman spaces A^p_α on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, where $0 < p < \infty$ and $\alpha > -1$. We extend this study in a very natural way to the case where α is *any* real number and $0 < p \leq \infty$. This unified treatment covers all classical Bergman spaces, Besov spaces, Lipschitz spaces, the Bloch space, the Hardy space H^2, and the so-called Arveson space. Some of our results about integral representations, complex interpolation, coefficient multipliers, and Carleson measures are new even for the ordinary (unweighted) Bergman spaces of the unit disk.

Résumé **(Théorie des espaces de Bergman dans la boule unité de $\mathbb{C}^n$)**

Ces dernières années il y a eu un grand nombre de travaux sur les espaces de Bergman pondérés A^p_α sur la boule unité $\mathbb{B}_n$ de $\mathbb{C}^n$, où $0 < p < \infty$ et $\alpha > -1$. Nous étendons cette étude, de manière très naturelle, au cas où α est un nombre réel *quelconque* et $0 < p \leq \infty$. Ce traitement unifié couvre tous les espaces de Bergman classiques, les espaces de Bésov, de Lipschitz, l'espace de Bloch, l'espace H^2 de Hardy, et celui appelé espace d'Arveson. Certains de nos résultats autour de la représentation entière, de l'interpolation complexe, des multiplicateurs de coefficients et des mesures de Carleson, sont nouveaux, y compris pour les espaces de Bergman ordinaires (non-pondérés) sur le disque unité.

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CHAPTER 1

INTRODUCTION

Throughout the paper we fix a positive integer n and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n dimensional complex Euclidean space. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n, \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

We use $H(\mathbb{B}_n)$ to denote the space of all holomorphic functions in $\mathbb{B}_n$.

For any $-\infty < \alpha < \infty$ we consider the positive measure

$$\mathrm{d}v_\alpha(z) = (1 - |z|^2)^\alpha \mathrm{d}v(z),$$

where $\mathrm{d}v$ is volume measure on $\mathbb{B}_n$. It is easy to see that $\mathrm{d}v_\alpha$ is finite if and only if $\alpha > -1$. When $\alpha > -1$, we normalize $\mathrm{d}v_\alpha$ so that it is a probability measure.

Bergman spaces with standard weights are defined as

$$A^p_\alpha = H(\mathbb{B}_n) \cap L^p(\mathbb{B}_n, \mathrm{d}v_\alpha),$$

where $p > 0$ and $\alpha > -1$. Here the assumption that $\alpha > -1$ is essential, because the space $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$ does not contain any holomorphic function other than 0 when $\alpha \le -1$. When $\alpha = 0$, we use A^p to denote the ordinary unweighted Bergman spaces. Bergman spaces with standard weights on the unit ball have been studied by numerous authors in recent years. See Aleksandrov [**2**], Beatrous-Burbea [**11**], Coifman-Rochberg [**21**], Rochberg [**46**], Rudin [**47**], Stoll [**57**], and Zhu [**71**] for results and references.

In this paper we are going to extend the definition of A^p_α to the case in which α is any real number and develop a theory for the extended family of spaces. More specifically, we study the following topics about the generalized spaces A^p_α: various characterizations, integral representations, atomic decomposition, complex interpolation, optimal pointwise estimates, duality, reproducing kernels when $p = 2$, Carleson type measures, and various special cases. A few of these are straightforward consequences or generalizations of known results in the case $\alpha > -1$ (we included them here with full proofs for the sake of a complete and coherent theory), thanks to the isomorphism between A^p_α and A^p via fractional integral and differential operators, while most others require new techniques and reveal new properties. Several of our results are new even in the case of ordinary Bergman spaces of the unit disk.

Our starting point is the observation that, for $p > 0$ and $\alpha > -1$, a holomorphic function f in $\mathbb{B}_n$ belongs to A^p_α if and only if the function $(1 - |z|^2)Rf(z)$ belongs to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$, where

$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z)$$

is the radial derivative of f. This result is well known to experts in the field and is sometimes referred to as a theorem of Hardy and Littlewood (especially in the one-dimensional case). See Beatrous **[9]**, Pavlovic **[42]**, or Theorem 2.16 of Zhu **[71]**. More generally, we can repeatedly apply this result and show that, for any positive integer k, a holomorphic function f is in A^p_α if and only if the function $(1-|z|^2)^k R^k f(z)$ belongs to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$.

Now for $p > 0$ and $-\infty < \alpha < \infty$ we fix a nonnegative integer k with $pk + \alpha > -1$ and define A^p_α as the space of holomorphic functions f in $\mathbb{B}_n$ such that the function $(1 - |z|^2)^k R^k f(z)$ belongs to $L^p(\mathbb{B}_n, \mathrm{d}v_\alpha)$. As was mentioned in the previous paragraph, this definition of A^p_α is consistent with the traditional definition when $\alpha > -1$. Also, it is easy to show (see Section 4) that the definition of A^p_α is independent of the integer k.

We also study a companion family of spaces defined using the sup-norm of a combination of powers of $1-|z|^2$ and partial derivatives of a holomorphic function f in $\mathbb{B}_n$. More specifically, for any real α we define Λ_α to be the space of holomorphic functions f in $\mathbb{B}_n$ such that the function $(1-|z|^2)^{k-\alpha}R^k f(z)$ is bounded in $\mathbb{B}_n$, where k is any nonnegative integer with $k > \alpha$. We are going to call them holomorphic Lipschitz spaces. Once again, it can be shown that the definition of Λ_α is independent of the choice of the integer k.

The two families of spaces A^p_α and Λ_α, with $0 < p < \infty$ and α real, cover any space (except H^∞ and its equivalents) of holomorphic functions that is defined in terms of membership in $L^p(\mathbb{B}_n, \mathrm{d}v)$, $0 < p \le \infty$, for any combination of partial derivatives of f and powers of $1 - |z|^2$. These spaces have appeared before in the literature under different names. For example, for any positive p and real s there is the classical diagonal Besov space B^s_p consisting of holomorphic functions f in $\mathbb{B}_n$ such that $(1-|z|^2)^{k-s}R^k f(z)$ belongs to $L^p(\mathrm{d}v_{-1})$, where k is any positive integer greater