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Bruno DEMANGE

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Bruno Demange

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B. Demange

Institut Fourier – UMR 5582, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères, France.

E-mail : Bruno.Demange@ujf-grenoble.fr

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Abstract. – This volume is devoted to several generalisations of the classical Hardy uncertainty principle on Euclidian spaces. Instead of comparing functions and their Fourier transforms a Gaussian, we compare them to the exponential of general non-degenerate quadratic forms, like for example the Lorentz form. Using the Bargmann transform, we translate the problem into the description of several classes of analytic functions of several variables, and at the same time simplify and unify proofs of results presented in several previous papers.

Résumé **(Principes d'incertitude associés à des formes quadratiques non dégénérées)**
Ce volume est consacré a des géneralisations du principe d'incertitude classique de Hardy dans les espaces Euclidiens. Au lieu de comparer les fonctions à des gaussiennes, nous les comparons a l'exponentielle de formes quadratiques non dégénérées, par exemple à la forme de Lorentz. Nous transformons ces problèmes à l'aide de la transformée de Bargmann, en des problèmes de description de certaines classes de fonctions entières de plusieurs variables. Ces méthode améliorent et simplifient des résultats publiés dans des travaux précédents.

CONTENTS

INTRODUCTION

This volume concerns certain forms of the uncertainty principle in harmonic analysis. The uncertainty principle is a general term for theorems that show that if a function f on $\mathbb{R}^d$ and its Fourier transform $\widehat{f}$ approximate g and $\widehat{g}$, then they must be equal.

The history of the uncertainty principle goes back to Heisenberg inequality of quantum mechanics, namely

$$\int |x|^2|\widehat{f}(x)|^2\,dx \times \int |\xi|^2|\widehat{f}(\xi)|^2\,d\xi \geq \frac{d^2}{16\pi^2}\|f\|_{L^2}^4,$$

where d is the dimension, and $\widehat{f}(y) = \int f(x)\exp(-2i\pi xy)\,dx$. This inequality is well known as the fact that the product of uncertainties of the position and the momentum is bounded below by an explicit constant, that involves the Planck constant. Equality occurs only for the Gaussian functions $f(x) = C\exp(-t|x|^2)$, $t > 0$.

The Hardy uncertainty principle [**13**] precised this unique property of Gaussian functions: if γ is a Gaussian function, there is no function f such that $|f| \leq \gamma$ and $|\widehat{f}| \leq \widehat{\gamma}$, except for the function γ itself (or its multiples). Variants of this results were proved by Morgan [**22**], Cowling-Price [**10**], not to mention the work that has been done on Lie Groups. This was illustrated more recently by a lost result of Beurling [**16**]:

$$\int_{\mathbb{R}} |f(x)||\widehat{f}(y)|\exp(2\pi|xy|)\,dx\,dy < \infty$$

implies that $f = 0$, while Gaussian functions make this integral finite when 2π is replaced by a smaller constant. This has been completed in [**7**], and one actually has, as a corollary, the following version of Hardy uncertainty principle: if

$$|f(x)\widehat{f}(y)| \leq \exp(-2\pi|xy|)$$

then f is a Gaussian function. In this example, we see that we can ask functions to decrease exponentially in some directions, and not in other, and still get an uncertainty principle.

This paper is essentially about the study of functions satisfying estimates of the form

$$|f(x)| \leq \exp(-\pi|q(x)|),\ |\widehat{f}(\xi)| \leq \exp(-\pi|q'(\xi)|), \tag{0.1}$$

where q and q' are two quadratic forms. We ask for an exponential decrease in some directions, but not in regions close to the isotropic sets of the forms, where they vanish. The classical Hardy uncertainty principle corresponds to positive quadratic forms. Take for example as previously the case of quadratic forms on $\mathbb{R}^2$ defined by $q(x,y) = 2xy$ and $q'(\xi,\eta) = 2\xi\eta$. We ask for

$$|f(x,y)| \leq \exp(-2\pi|xy|), \ |\widehat{f}(\xi,\eta)| \leq \exp(-2\pi|\xi\eta|). \tag{0.2}$$

Here q and q' are not positive, and we cannot expect solutions to be integrable. Take for example

$$f(x,y) = \operatorname{sgn}(x)\exp(-2\pi|xy|).$$

It is not in any L^p space except L^∞. However, in the distribution sense, we have

$$\widehat{f}(\xi,\eta) = -i\operatorname{sgn}(\xi)\exp(-2\pi|\xi\eta|),$$

so that (0.2) is satisfied.

We see with this example that studying solutions of (0.1) requires to work on the level of distribution. In this setting, (0.1) can be rewritten in the following: study distributions f in the Schwartz space S' so that

$$f(\cdot)\exp(\pm\pi q(\cdot)) \in S', \ \widehat{f}(\cdot)\exp(\pm\pi q'(\cdot)) \in S'. \tag{0.3}$$

When q and q' are both positive quadratic forms, this corresponds to the classical Hardy uncertainty principle, except that it is stated in a distributional setting. In the simplest case, the conditions are

$$f(\cdot)\exp(\pi|\cdot|^2) \in S', \ \widehat{f}(\cdot)\exp(\pi|\cdot|^2) \in S'. \tag{0.4}$$

To solve this problem, we had to work with more regular objects than distributions. We do this using the Bargmann transform, which is essentially a convolution with a Gaussian function. If f is a tempered distribution, its Bargmann transform is the entire function defined by

$$\mathscr{B}(f)(z) = \exp\Big(\frac{\pi}{2}z^2\Big) f \star \gamma(z),$$

where $\gamma(x) = \exp(-\pi|x|^2)$. It has been introduced by Bargmann in [**3, 4**].

We already used the Bargmann transform in [**7**], even if not explicitly. There we studied functions satisfying Beurling type conditions, of the form

$$(1+|x|+|y|)^{-N} f(x)\widehat{f}(y)\exp(2\pi|xy|) \in L^1. \tag{0.5}$$

Even if it was the scheme of Hörmander's proof for Beurling theorem, regularity of f is not a direct consequence of (0.5), while Hardy's conditions imply directly that f extends to an entire function of order 2. Our trick was to convolve f with γ. Since f has to be a Hermite function, so does $g = f \star \gamma$. We showed that the new function g satisfies also (0.5). This is the Bargmann transform of f, up to a factor.