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A FUNDAMENTAL DOMAIN FOR V_3

Mary Rees

Abstract. – We describe a fundamental domain for the punctured Riemann surface $V_{3,m}$ which parametrises (up to Möbius conjugacy) the set of quadratic rational maps with numbered critical points, such that the first critical point has period three, and such that the second critical point is not mapped in m iterates or less to the periodic orbit of the first. This gives, in turn, a description, up to topological conjugacy, of all dynamics in all type III hyperbolic components in V_3, and gives indications of a topological model for V_3, together with the hyperbolic components contained in it.

***Résumé* (Un domaine fondamental pour V_3).** – Nous décrivons un domaine fondamental pour la surface de Riemann $V_{3,m}$ qui paramétrise (à conjugaison près) l'ensemble des fonctions rationelles par le biais des points critiques énumérés, de manière à ce que le premier point critique ait une période de 3, et que le deuxième point critique ne soit pas envoyé sur le premier après m itérations ou moins. Cela nous fournit une description, à conjugaison topologique près, des dynamiques de toutes les composantes de type III en V_3, et nous donne des indications sur un modèle topologique de V_3, au même temps que l'ensemble des composantes hyperboliques qui y sont contenues.

CONTENTS

CHAPTER 1

INTRODUCTION

In this paper, we give a complete description of half the hyperbolic components in a certain parameter space of quadratic rational maps. The parameter space is known as V_3, and consists of all maps

$$h_a : z \mapsto \frac{(z-a)(z-1)}{z^2}, \quad a \in \mathbb{C}, \ a \neq 0.$$

This map has two critical points,

$$c_1 = 0, \ c_2 = c_2(a) = \frac{2a}{a+1},$$

and c_1 is periodic of period 3, with orbit

$$0 \mapsto \infty \mapsto 1 \mapsto 0.$$

There is thus one free critical point, c_2. Describing the hyperbolic components in V_3 — and in fact only half, or in numerical terms, two thirds of them – might seem rather a modest project, but has, in fact, been ongoing for some twenty years, and has generated an extensive theory, which will be summarized, as far as it concerns us here, in Section 4. It is essentially because of technical difficulties in the development of the theory that only half the hyperbolic components in V_3 are described. A conjectural picture for the remaining half is not too hard to obtain. Remarks are made on this at various points, but no explicit description is given.

One can obviously ask why V_3 has been chosen. The space V_n is the space of quadratic rational maps with one critical point of period n, quotiented by Möbius conjugacy. A quadratic rational map with one fixed critical point is Möbius conjugate to a quadratic polynomial. Any quadratic polynomial is affinely conjugate to one of the form

$$f_c : z \mapsto z^2 + c$$

for some $c \in \mathbb{C}$, and this, thus, is the space V_1. This parameter space must be one of the most studied in all dynamics. There are several reasons for this. One is that dynamics within this parameter space is rich and varied. Another is that there is a pretty full description of the variation of dynamics within this parameter space, at least up to topological conjugacy, and at least conjecturally. Another is that this conjectural description of the dynamics, which is very detailed, but not fully complete, and which includes a conjecture on the topology of the Mandelbrot set, is still not proved, although there has been very interesting progress recently. The missing piece has analogues in the theory of Kleinian groups — where the corresponding problem has been solved [**3, 4, 5, 8, 9, 26, 33**] — not to mention other, less well-understood, parameter spaces of holomorphic maps. So there are attractively simple questions one can ask, even in this case, which have very detailed, but not complete, answers, and one hopes that this case will shed some light on other parameter spaces of holomorphic maps.

Actually, the parameter space of quadratic polynomials differs from most other parameter spaces in one very fundamental respect. There is a natural "base" map in the space of quadratic polynomials, namely, the map $f_0 : z \mapsto z^2$, and for any hyperbolic map f (and hopefully some nonhyperbolic also) in the connectedness locus — also called the Mandelbrot set — there is an essentially unique description of the dynamics of f in terms of f_0. This is because the complement of the Mandelbrot set is simply connected, and the Mandelbrot set itself has a natural tree-like structure. There appears to be essentially one natural path from f_0 to f within the Mandelbrot set. This is very far from being the case in other parameter spaces of quadratic rational maps, although the structure of V_2 is quite simple, as we shall indicate — but not prove — in Section 2.

For $n \geq 3$, V_n identifies with the family of quadratic rational maps

$$f_{c,d} : z \mapsto 1 + \frac{c}{z} + \frac{d}{z^2}$$

($d \neq 0$) for which the critical point 0 is constrained to have period n. If $n \geq 3$, then the maps $f_{c,d} \in V_n$ are in one-to-one correspondence with Möbius conjugacy classes of quadratic rational maps f with named critical point $c_1(f)$ of period n, where we use only conjugacies which map $c_1(f)$ to 0. These parameter spaces were the main objects of study in [**29, 30, 31**]. The main point of [**29, 30**] was that it is possible to describe dynamics of hyperbolic maps $f_{c,d}$ in V_n in terms of a path from some fixed base map g_0 - and, of course, in terms of that base map g_0. In [**29**] we used a polynomial (up to Möbius conjugacy) as base, and the resulting theorem was called the *Polynomial-and-Path Theorem*. It was shown that a path in parameter space from g_0 to $f_{c,d}$ gave rise to a path in the dynamical plane of g_0. So the theorem showed how to define $f_{c,d}$ in terms of g_0 and the path in the dynamical plane of g_0. The main problems with such a result were, firstly, identifying which paths in the dynamical plane of g_0 were associated with paths in V_n, and therefore, with hyperbolic maps in V_n, and secondly, to determine when two paths gave rise to a description of the