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# ENTROPY OF MEROMORPHIC MAPS AND DYNAMICS OF BIRATIONAL MAPS

Henry DE THÉLIN & Gabriel VIGNY

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# ENTROPY OF MEROMORPHIC MAPS AND DYNAMICS OF BIRATIONAL MAPS

**Henry De Thélin, Gabriel Vigny**

***Abstract.*** – We study the dynamics of meromorphic maps for a compact Kähler manifold $X$. More precisely, we give a simple criterion that allows us to produce a measure of maximal entropy. We can apply this result to bound the Lyapunov exponents.

Then, we study the particular case of a family of generic birational maps of $\mathbb{P}^k$ for which we construct the Green currents and the equilibrium measure. We use for that the theory of super-potentials. We show that the measure is mixing and gives no mass to pluripolar sets. Using the criterion we get that the measure is of maximal entropy. It implies finally that the measure is hyperbolic.

***Résumé*** **(Entropie des applications méromorphes et dynamique des applications birationnelles)**

On étudie la dynamique des applications méromorphes sur les variétés kählériennes compactes. Plus précisément, on donne un critère simple qui permet de produire des mesures d'entropie maximale. On peut appliquer ce résultat pour borner les exposants de Lyapounov.

Ensuite, on étudie le cas particulier d'une famille générique d'applications birationnelles de $\mathbb{P}^k$ pour laquelle on construit les courants de Green et la mesure d'équilibre. On utilise pour cela la théorie des super-potentiels. On montre que la mesure est mélangeante et qu'elle n'a pas de masse sur les ensembles pluripolaires. En utilisant le critère on obtient que la mesure est d'entropie maximale. Cela implique finalement que la mesure est hyperbolique.

# CONTENTS

# CHAPTER 1

# INTRODUCTION

Let $X$ be a complex compact Kähler manifold of complex dimension $k$ endowed with a Kähler form $\omega$. We consider $f : X \to X$ a dominant meromorphic map and we denote by $I$ its indeterminacy set. We want to study the dynamics of $f$ in other words the behavior of the sequence of iterates $(f^n)$. We are particularly interested in the study of the ergodic properties of $f$.

Such study often starts with computing the topological entropy which gives an insight into how complicated the dynamics is. Classical objects in that case are ergodic measures of maximal entropy. Indeed, the support of such measure $\mu$ is an invariant set where the complexity of the dynamics is maximal. On this support, the dynamics is well understood in a statistical sense by Birkhoff's ergodic theorem as almost every orbit is equidistributed for the measure $\mu$. One can then study finer properties of the measure: mixing, speed of mixing, dimension of the measure ...

In order to understand more precisely the dynamics near a point $x$ in the support of the measure, one classicaly tries to compute the Lyapunov exponents. Roughly speaking, if no Lyapunov exponent is zero, these are numbers which give a rate of contraction and a rate of expansion in some stable and unstable manifolds. In other words, on the stable manifold the orbit of a point tends to the orbit of $x$ at a speed given by some negative Lyapunov exponent and on the unstable manifold the backward orbit of a point tends to the backward orbit of $x$ at a speed given by some positive Lyapunov exponent. When no Lyapunov exponent is zero, the measure is said to be *hyperbolic*. Finding examples of hyperbolic measures is a central question in dynamics and complex dynamics provides usually many of those (see **[45]** for definitions and results on hyperbolic measures).

In the particular case of complex dynamics, the topological entropy is related to the *dynamical degrees*. For $l = 0 \dots k$, we write:

$$\lambda_l(f) := \int_X f^*(\omega^l) \wedge \omega^{k-l}.$$

The $l$-th *dynamical degree* of $f$ is defined by (see **[48]** and **[22]**):

$$d_l := \lim_{n\to+\infty} (\lambda_l(f^n))^{1/n}.$$

The degree $d_l$ measures the asymptotic spectral radius of the action of $f^*$ on the cohomology group $H^{l,l}(X)$. The last degree $d_k$ is the *topological degree*. It can be shown that the sequence of degrees is increasing up to a rank $s$ and then it is decreasing (see **[40]**). These quantities are of algebraic nature and there is a bound from above of the topological entropy by $\max_{0\le s\le k}\log d_s$ (see **[41]** for the holomorphic case and **[25]**, **[22]** for the meromorphic case).

When one of the dynamical degree is strictly higher than the others, $f$ is called *cohomologically hyperbolic*. In that case, it is expected that there exists a measure of maximal entropy $\max_{0\le s\le k}\log d_s$ (see **[43]**). Such measure is expected to be hyperbolic (and the saddle points are expected to be equidistributed along the measure). Recall the first author's result on Lyapunov exponents (Corollary 3 in **[11]**) when $f$ is cohomologically hyperbolic: if one can find a measure $\mu$ of entropy $\max_{0\le s\le k}\log d_s$, then it is hyperbolic with estimates on the Lyapunov exponents provided that $\log dist(x, I\cup C) \in L^1(\mu)$ ($I$ is the indeterminacy set of $f$ and $C$ the critical set). In particular, in order to find hyperbolic measures, it is enough to find measures of maximal entropy.

This will be our aim in the first part of this article (Chapter 2). Following Yomdin's approach (**[55]**), we give a criterion that allows us to produce invariant measures of maximal entropy for a meromorphic map on a compact Kähler manifold $X$. In a second part (Chapter 3), we study the more precise case of a family of birational maps of $\mathbb{P}^k$ for which we construct the equilibrium measure. We show that it is mixing and using the results of the first part we show that it is of maximal entropy. In particular the measure is hyperbolic. Let us detail our results.

Let $d$ denote the distance in $X$ and recall that $I$ is the indeterminacy set of $f$. In Chapter 2, we do not assume that $f$ is cohomologically hyperbolic. Chapter 2 is devoted to the proof of the following theorem:

THEOREM 1. – *Consider the sequence of measures:*

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} f^i_*\left(\frac{(f^n)^*\omega^l\wedge\omega^{k-l}}{\lambda_l(f^n)}\right).$$

*Assume that there exists a converging subsequence* $\mu_{\psi(n)}\to\mu$ *with:*

$$(H):\ \lim_{n\to+\infty}\int \log d(x,I)\,d\mu_{\psi(n)}(x) = \int \log d(x,I)\,d\mu(x) > -\infty.$$

*Then* $\mu$ *is an invariant measure of metric entropy larger than or equal to* $\log d_l$.