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PROJECTIONS IN SEVERAL COMPLEX VARIABLES

Chin-Yu Hsiao

Abstract. – This work consists two parts. In the first part, we completely study the heat equation method of Menikoff-Sjöstrand and apply it to the Kohn Laplacian defined on a compact orientable connected CR manifold. We then get the full asymptotic expansion of the Szegő projection for $(0, q)$ forms when the Levi form is non-degenerate. This generalizes a result of Boutet de Monvel and Sjöstrand for $(0, 0)$ forms. Our main tools are Fourier integral operators with complex valued phase Melin and Sjöstrand functions.

In the second part, we obtain the full asymptotic expansion of the Bergman projection for $(0, q)$ forms when the Levi form is non-degenerate. This also generalizes a result of Boutet de Monvel and Sjöstrand for $(0, 0)$ forms. We introduce a new operator analogous to the Kohn Laplacian defined on the boundary of a domain and we apply the heat equation method of Menikoff and Sjöstrand to this operator. We obtain a description of a new Szegő projection up to smoothing operators. Finally, we get our main result by using the Poisson operator.

Résumé **(Projecteurs en plusieurs variables complexes).** – Ce travail comporte deux parties. Dans la première, nous appliquons la méthode de Menikoff-Sjöstrand au laplacien de Kohn, défini sur une varieté CR compacte orientée connexe et nous obtenons un développement asymptotique complet du projecteur de Szegő pour les $(0, q)$ formes quand la forme de Levi est non-dégénérée. Cela généralise un résultat de Boutet de Monvel et Sjöstrand pour les $(0, 0)$ formes. Nous utilisons des opérateurs intégraux de Fourier à phases complexes de Melin et Sjöstrand.

Dans la deuxième partie, nous obtenons un développement asymptotique de la singularité du noyau de Bergman pour les $(0, q)$ formes quand la forme de Levi est non-dégénérée. Cela généralise un résultat de Boutet de Monvel et Sjöstrand pour les $(0, 0)$ formes. Nous introduisons un nouvel opérateur analogue au laplacien de Kohn défini sur le bord du domaine, et nous y appliquons la méthode de Menikoff-Sjöstrand. Cela donne une description modulo les opérateurs régularisants d'un nouvel projecteur de Szegő. Enfin, nous obtenons notre résultat principal en utilisant l'opérateur de Poisson.

CONTENTS

INTRODUCTION

The Bergman and Szegő projections are classical subjects in several complex variables and complex geometry. By Kohn's regularity theorem for the $\overline{\partial}$-Neumann problem (1963, [**23**]), the boundary behavior of the Bergman kernel is highly dependent on the Levi curvature of the boundary. The study of the boundary behavior of the Bergman kernel on domains with positive Levi curvature (strictly pseudoconvex domains) became an important topic in the field then. In 1965, L. Hörmander [**13**] determined the boundary behavior of the Bergman kernel. C. Fefferman (1974, [**9**]) established an asymptotic expansion at the diagonal of the Bergman kernel. More complete asymptotics of the Bergman kernel was obtained by Boutet de Monvel and Sjöstrand (1976, [**34**]). They also established an asymptotic expansion of the Szegő kernel on strongly pseudoconvex boundaries. All these developments concerned pseudoconvex domains. For the nonpseudoconvex domain, there are few results. R. Beals and P. Greiner (1988, [**1**]) proved that the Szegő projection is a Heisenberg pseudodifferential operator, under certain Levi curvature assumptions. Hörmander (2004, [**19**]) determined the boundary behavior of the Bergman kernel when the Levi form is negative definite by computing the leading term of the Bergman kernel on a spherical shell in $\mathbb{C}^n$.

Other developments recently concerned the Bergman kernel for a high power of a holomorphic line bundle. D. Catlin (1997, [**5**]) and S. Zelditch (1998, [**38**]) adapted a result of Boutet de Monvel-Sjöstrand for the asymptotics of the Szegő kernel on a strictly pseudoconvex boundary to establish the complete asymptotic expansion of the Bergman kernel for a high power of a holomorphic line bundle with positive curvature. X. Dai, K. Liu and X. Ma (2004, [**7**]) obtained the full off-diagonal asymptotic expansion and Agmon estimates of the Bergman kernel for a high power of positive line bundle by using the heat kernel method. Recently, a new proof of the existence of the complete asymptotic expansion was obtained by B. Berndtsson, R. Berman and J. Sjöstrand (2004, [**2**]) and by X. Ma, G. Marinescu (2004, [**27**]). Without the positive curvature assumption, R. Berman and J. Sjöstrand (2005, [**3**]) obtained a full asymptotic expansion of the Bergman kernel for a high power of a line bundle when the curvature is non-degenerate. The approach of Berman and Sjöstrand builds on the heat equation method of Menikoff-Sjöstrand (1978, [**29**]). The expansion was obtained independently by X. Ma and G. Marinescu (2006, [**25**], without a phase function) by using a spectral gap estimate for the Hodge Laplacian. The main analytic tool of

X. Ma and G. Marinescu is the analytic localization technique in local index theory developed by Bismut-Lebeau. We refer the readers to the very nice book ([**26**]) by X. Ma and G. Marinescu for this approach.

Recently, Hörmander (2004, [**19**]) studied the Bergman projection for $(0, q)$ forms. In that paper (page 1306), Hörmander suggested: "A carefull microlocal analysis along the lines of Boutet de Monvel-Sjöstrand should give the asymptotic expansion of the Bergman projection for $(0, q)$ forms when the Levi form is non-degenerate."

The main goal for this work is to achieve Hörmander's wish-more precisely, to obtain an asymptotic expansion of the Bergman projection for $(0, q)$ forms. The first step of my research is to establish an asymptotic expansion of the Szegő projection for $(0, q)$ forms. Then, find a suitable operator defined on the boundary of domain which plays the same role as the Kohn Laplacian in the approach of Boutet de Monvel-Sjöstrand.

This work consists two parts. In the first paper, we completely study the heat equation method of Menikoff-Sjöstrand and apply it to the Kohn Laplacian defined on a compact orientable connected CR manifold. We then get the full asymptotic expansion of the Szegő projection for $(0, q)$ forms when the Levi form is non-degenerate. We also compute the leading term of the Szegő projection.

In the second paper, we introduce a new operator analogous to the Kohn Laplacian defined on the boundary of a domain and we apply the method of Menikoff-Sjöstrand to this operator. We obtain a description of a new Szegő projection up to smoothing operators. Finally, by using the Poisson operator, we get the full asymptotic expansion of the Bergman projection for $(0, q)$ forms when the Levi form is non-degenerate.

These two papers can be read independently. We hope that this work can serve as an introduction to certain microlocal techniques with applications to complex geometry and CR geometry.

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