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REPULSION FROM RESONANCES

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REPULSION FROM RESONANCES

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Abstract. – We consider slow-fast systems with periodic fast motion and integrable slow motion in the presence of both weak and strong resonances. Assuming that the initial phases are random and that appropriate non-degeneracy assumptions are satisfied we prove that the effective evolution of the adiabatic invariants is given by a Markov process. This Markov process consists of the motion along the trajectories of a vector field with occasional jumps. The generator of the limiting process is computed from the dynamics of the system near strong resonances.

Résumé **(Répulsion par les résonances).** – Nous considérons des systèmes « lents-rapides », dont le mouvement rapide est périodique et le mouvement lent intégrable, en présence de résonances faibles ou fortes. En supposant que les phases initiales sont aléatoires et que certaines conditions de non-dégénérescence sont satisfaites, nous démontrons que l'évolution effective des invariants adiabatiques est donnée par un processus de Markov. Ce processus de Markov consiste en un mouvement le long des trajectoires d'un champ de vecteurs qui peut présenter des sauts occasionnels. Le générateur du processus limite est calculé à partir de la dynamique du système au voisinage des résonances fortes.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Introduction

Averaging method is one of the most classical and most effective tools in dynamics. The basic idea is very simple. Consider a two scale system

$$\dot{y} = Y(x, y, \varepsilon), \quad \dot{x} = \frac{X(x, y, \varepsilon)}{\varepsilon}, \quad \text{where } \varepsilon \ll 1.$$

If we are interested in the evolution of the slow variables y the direct computations are costly since y changes at times $\mathcal{O}(1)$ while any numerical procedure should have step $o(\varepsilon)$ which is the natural time scale for the change of fast variables. In fact, the instabilities of the fast system can make computations unreliable. Therefore it is natural to approximate y by the solution of the effective equation

$$\dot{\bar{y}} = \overline{Y}(\bar{y}) \quad \text{where} \quad \overline{Y}(\bar{y}) = \big\langle Y(\bar{y}, x, 0)\big\rangle \tag{1.1}$$

and $\langle\, . \,\rangle$ denotes the averaging with respect to an invariant measure for the frozen system

$$\dot{x} = X(x, \bar{y}, 0)$$

(finding the correct measure for $\langle\, . \,\rangle$ is part of the problem).

While the averaging method itself was invented about quarter of the millennium ago motivated by the needs of the Celestial Mechanics (see **[58]** for a historical survey) the work on its rigorous justification started much later. The first results were limited to the case where the fast motion is periodic **[20]**, **[38]**, **[9]**, **[36]** or more generally uniquely ergodic **[8]**. In the uniquely ergodic case there is only one invariant measure so the meaning of $\langle\, . \,\rangle$ in (1.1) is clear. However the uniquely ergodic setting is insufficient even for describing small perturbation of nearly integrable systems (because in this case the unperturbed system contains resonant tori which possess many invariant measures). The justification of averaging method in the general case is more subtle since in that case the actual trajectory is close to the averaged one not everywhere but only on a large measure set of initial conditions. The work on the justification of the averaging method in the general setting was undertaken in the second half of 20th century mostly by the Soviet School (see **[2]**, **[3]**, **[4]**, **[21]**, **[32]**, **[33]**, **[35]**, **[39]**, **[41]**, **[43]**). By now the averaging principle is justified under quite general conditions (see e.g. **[5]**, **[24]**, **[34]**, **[37]**).

In the case the fast motion is chaotic (for example an Anosov system or a Markov process) one can also obtain the limiting distribution for the difference between the actual and the averaged trajectory (see review **[34]**). Such estimates are not yet available in the classical setting of the quasiperiodic fast motion. One situation where such deviations are important is when the averaged system has first integrals (which are called *adiabatic invariants* for the original system). The change of the adiabatic invariants occurs only due to deviations from the averaged motion. It is well known that in the quasiperiodic case the main source of deviations from the averaged motion happens due to passages through resonances. The contribution of the individual resonance has been computed by several authors **[14]**, **[26]**, **[27]**, **[42]**, **[41]**, **[48]**. There are many examples of the systems where multiple resonance passages can lead to destruction of adiabatic invariants on the appropriate time scale (see the above cited papers as well as **[29]**, **[31]**, **[40]**, **[47]**, **[57]**, **[59]**, **[60]**, **[61]**, **[62]**). Since the quasiperiodic case remains the prevalent source of application of the averaging techniques the development of the statistical theory of adiabatic invariants is one of the most important problems in the averaging theory. The goal of the present paper is to make a step in this direction by considering the simplest case of periodic fast motion.

The basic idea is that the passage through resonances makes the dynamics hyperbolic on most of the phase space **[47]**. In fact the hyperbolicity is created due to the combination of strong shearing away from resonances with the destruction of the shear-invariant foliation near the resonances. (For the general discussion of the sheared induced stochasticity we refer the reader to **[1]**, **[28]**, **[55]**, **[63]**.) Therefore the methods developed to treat hyperbolic fast motion (see **[7]**, **[13]**, **[17]**, **[18]**, **[19]**) can be applied. The difference between our approach and the other papers on quasiperiodic averaging is that rather than computing C^0-norm of the deviation with a very high precision we get more coarse information about C^2-norms and exploit the properties which are shared by our system and its C^2-small perturbations. Unfortunately this shift of the point of view leads to the increased size of the paper. Indeed the C^2-estimates required for our method were not readily available in the literature (even though their derivation proceeds similarly to the C^0-bounds). For completeness we provide the required estimates in the appendices.

We hope that this new point of view can be useful in the general quasiperioidic case. However new ideas will be required to handle the overwhelming growth of complexity coming from the fact that in the quasiperiodic case there are infinitely many resonances.