Mémoires
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

Numéro 130-131
Nouvelle série

Yukiyoshi NAKKAJIMA

2 0 1 2

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre National de la Recherche Scientifique

Comité de rédaction

Jean BARGE
Gérard BESSON
Emmanuel BREUILLARD
Antoine CHAMBERT-LOIR
Jean-François DAT
Jean-Marc DELORT
Charles FAVRE
Daniel HUYBRECHTS
Yves LE JAN
Laure SAINT-RAYMOND
Wilhelm SCHLAG
Raphaël KRIKORIAN (dir.)

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 60 € ($ 90)
Abonnement Europe : 262 € hors Europe : 296 € ($ 444)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën
Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

© *Société Mathématique de France 2012*

Tous droits réservés (article L 122–4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335–2 et suivants du CPI.

ISSN 0249-633-X

ISBN 978-2-85629-376-8

Directeur de la publication : Marc PEIGNÉ

MÉMOIRES DE LA SMF 130/131

WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

Yukiyoshi Nakkajima

Société Mathématique de France 2012
Publié avec le concours du Centre National de la Recherche Scientifique

Y. Nakkajima
Department of Mathematics, Tokyo Denki University,
5 Senju-Asahi-cho Adachi-ku, Tokyo 120-8551, Japan.
E-mail : nakayuki@cck.dendai.ac.jp

2000 *Mathematics Subject Classification*. — 14F30.

***Key words and phrases*. —** Rigid cohomology, Log crystalline cohomology, Log de Rham-Witt complex, Weights, Slopes.

The author is supported from JSPS Grant-in-Aid for Scientific Research (C) (Grant No. 20540025). He is also supported by JSPS Core-to-Core Program 18005 whose representative is Makoto Matsumoto.

WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

Yukiyoshi Nakkajima

***Abstract.* —** We construct a theory of weights on the rigid cohomology of a separated scheme of finite type over a perfect field of characteristic $p > 0$ by using the log crystalline cohomology of a split proper hypercovering of the scheme. We also calculate the slope filtration on the rigid cohomology by using the cohomology of the log de Rham-Witt complex of the hypercovering.

***Résumé.* —** Nous construisons une théorie des poids sur la cohomologie rigide d'un schéma séparé de type fini sur un corps parfait de caractéristique $p > 0$ en utilisant la cohomologie log-cristalline d'un hyperrecouvrement propre scindé du schéma. Nous calculons aussi la filtration par les pentes sur la cohomologie rigide en utilisant la cohomologie du complexe de de Rham-Witt logarithmique de l'hyperrecouvrement.

© Mémoires de la Société Mathématique de France 130/131, SMF 2012

CONTENTS

1. Introduction

J.-P. Serre has conjectured the existence of the virtual Betti number of a separated scheme of finite type over a field (see [**33**]). Furthermore, A. Grothendieck has conjectured the existence of, so to speak, the virtual slope number and the virtual Hodge number of the scheme (see [*loc. cit.*]). Let us recall the numbers briefly as follows.

Let κ be an algebraically closed field of characteristic $p \geq 0$. Let $\mathbf{S}(\kappa)$ be the set of isomorphism classes of separated schemes of finite type over κ. For a separated scheme U of finite type over κ, we denote by $[U]$ the element of $\mathbf{S}(\kappa)$ defined by U. For a function $f\colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$, we denote $f([U])$ by $f(U)$ for simplicity of notation. Let r be a nonnegative integer. Then Serre has conjectured that there exists a unique function

$$h_{\mathrm{c}}^r\colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$$

satisfying the following two properties (see [**33**]):

▷ For a closed subscheme Z of U,

$$h_{\mathrm{c}}^r(U) = h_{\mathrm{c}}^r(Z) + h_{\mathrm{c}}^r(U \setminus Z). \tag{I-a}$$

▷ If U is a proper smooth scheme over κ, then

$$h_{\mathrm{c}}^r(U) = (-1)^r \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^r(U, \mathbb{Q}_\ell) \tag{I-b}$$

for a prime number $\ell \neq p$.

Here the letter 'c' in the notation h_{c}^r stands for the compact support. Heuristically $h_{\mathrm{c}}^r(U)$ is given by the formula (see [*loc. cit.*]):

$$h_{\mathrm{c}}^r(U) = \begin{cases} \sum_{m=0}^{\infty}(-1)^m \dim_{\mathbb{Q}_\ell} \mathrm{gr}_r^P H_{\text{ét},c}^m(U, \mathbb{Q}_\ell) & (p > 0), \\ \sum_{m=0}^{\infty}(-1)^m \dim_{\mathbb{Q}} \mathrm{gr}_r^P H_{\mathrm{c}}^m(U_{\mathrm{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}), \end{cases} \tag{1.0.1}$$

where P is the 'weight filtration' on $H_{\text{ét},c}^m(U, \mathbb{Q}_\ell)$ (resp. $H_{\mathrm{c}}^m(U_{\mathrm{an}}, \mathbb{Q})$). (For the cohomology without compact support, see (1.0.3) below for the definition of the weight filtration P when κ is $\overline{\mathbb{F}}_p$ (resp. $\mathbb{C}$).) Set

$$H_{\mathrm{c}}^m(U) = \begin{cases} H_{\text{ét},c}^m(U, \mathbb{Q}_\ell) & (p > 0), \\ H_{\mathrm{c}}^m(U_{\mathrm{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}). \end{cases}$$

Heuristically (I-a) is obtained from the strict exactness with respect to P of the exact sequence

$$\cdots \longrightarrow H_{\mathrm{c}}^m(U \setminus Z) \longrightarrow H_{\mathrm{c}}^m(U) \longrightarrow H_{\mathrm{c}}^m(Z) \longrightarrow \cdots .$$

Serre has called $h^r_c(U)$ the virtual Betti number of $[U]$. He has also remarked that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h^r_c. Serre's conjecture was the starting point of Grothendieck's theory of weights (see [**33**]).

In [**33**] Grothendieck has conjectured, for two nonnegative integers i and j, there exists a function

$$h^{ij}_c \colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$$

such that

$$h^r_c(U) = \sum_{i+j=r} h^{ij}_c(U). \tag{II-a}$$

In the case $p > 0$, let $\mathcal{W}$ be the Witt ring of κ, K_0 the fraction field of $\mathcal{W}$ and $\mathcal{W}\Omega^i_U$ the de Rham-Witt sheaf of U of degree i (see [**47**]). We call $h^{ij}_c(U)$ the *virtual slope number* and the *virtual Hodge number* of $[U]$ if $p > 0$ and $p = 0$, respectively, because we require that h^{ij}_c satisfies the equation:

$$h^{ij}_c(U) = \begin{cases} (-1)^{i+j}\dim_{K_0}(H^j(U, \mathcal{W}\Omega^i_U) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ (-1)^{i+j}\dim_\kappa H^j(U, \Omega^i_{U/\kappa}) & (p = 0) \end{cases} \tag{II-b}$$

if U is a proper smooth scheme over κ. We also require that h^{ij}_c satisfies the equation

$$h^{ij}_c(U) = h^{ij}_c(Z) + h^{ij}_c(U \setminus Z). \tag{II-c}$$

Furthermore, we conjecture that h^{ij}_c is uniquely determined by the equations (II-b) and (II-c). It is easy to check that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h^{ij}_c. The existence of h^{ij}_c $(i, j \in \mathbb{N})$ implies the existence of h^r_c $(r \in \mathbb{N})$ because (I-a) is clear by (II-a) and (II-c) and because (I-b) holds as follows: if U is a proper smooth scheme over κ, then we have the equations

$$\begin{aligned} \dim_{\mathbb{Q}_\ell} H^r_{\text{ét}}(U, \mathbb{Q}_\ell) &= \begin{cases} \dim_{K_0}(H^r_{\text{crys}}(U/\mathcal{W}) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ \dim_\kappa H^r_{\text{dR}}(U/\kappa) & (p = 0), \end{cases} \\ &= \begin{cases} \sum_{i+j=r} \dim_{K_0}(H^j(U, \mathcal{W}\Omega^i_U) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ \sum_{i+j=r} \dim_\kappa H^j(U, \Omega^i_{U/\kappa}) & (p = 0). \end{cases} \end{aligned} \tag{1.0.2}$$

The first (resp. the second) equation in (1.0.2) in the case $p > 0$ has been obtained in [**54**], [**16**] and [**69**] (resp. [**47**]). The first equation in (1.0.2) for