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# PERSISTENCE OF STRATIFICATIONS OF NORMALLY EXPANDED LAMINATIONS

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# PERSISTENCE OF STRATIFICATIONS OF NORMALLY EXPANDED LAMINATIONS

**Pierre Berger**

***Abstract.*** — This manuscript complements the Hirsch-Pugh-Shub (HPS) theory on persistence of normally hyperbolic laminations and implies several structural stability theorems.

We generalize the concept of lamination by defining a new object: the stratification of laminations. It is a stratification whose strata are laminations. The main theorem implies the persistence of some stratifications whose strata are normally expanded. The dynamics is a $C^r$-endomorphism of a manifold (which is possibly not invertible and with critical points). The persistence means that any $C^r$-perturbation of the dynamics preserves a $C^r$-close stratification.

If the stratification consists of a single stratum, the main theorem implies the persistence of normally expanded laminations by endomorphisms, and hence implies HPS theorem. Another application of this theorem is the persistence, as stratifications, of submanifolds with boundary or corners normally expanded. Several examples are also given in product dynamics.

As diffeomorphisms that satisfy axiom A and the strong transversality condition (AS) defines canonically two stratifications of laminations: the stratification whose strata are the (un)stable sets of basic pieces of the spectral decomposition. The main theorem implies the persistence of some "normally AS" laminations which are not normally hyperbolic and other structural stability theorems.

***Résumé.*** — Ce travail s'inscrit dans le prolongement de celui de Hirsch-Pugh-Shub (HPS) sur la persistance des laminations normalement hyperboliques, et implique plusieurs théorèmes de stabilité structurelle.

On généralise le concepte de lamination par une nouvelle catégorie d'objets : les stratifications de laminations. Il s'agit de stratifications, dont les strates sont des laminations. On propose alors un théorème assurant la persistance de certaines stratifications dont chaque strate est une lamination normalement dilatée. La dynamique est un $C^r$-endomorphisme d'une variété (qui n'est donc pas forcément inversible et qui peut avoir des points critiques). La persistance signifie que toute $C^r$-perturbation de la dynamique préserve une stratification $C^r$-proche.

Quand la stratification est formée d'une unique strate, le théoreme principal donne la persistance des laminations normalement dilatées par un endomorphisme, et implique ainsi le théorème de HPS. Une autre application de ce théorème est la persistance des variétés à bord ou à coins normalement dilatés. Beaucoup examples sont donnés facilement en dynamique produit.

Aussi les difféomorphismes vérifiant l'axiome A et la condition de transversalité forte (ATF) possèdent deux stratifications de laminations canoniques : celle dont les strates sont les ensembles stables (resp. instables) de ses pièces basiques. Ainsi, notre théorème implique la persistance de certaines laminations "normalement ATF" qui ne sont pas normalement hyperboliques et d'autres théorèmes de stabilité structurelle.

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# INTRODUCTION

## 0.1. Motivations

In 1977, M. Hirsch, C. Pugh and M. Shub [**15**] developed a theory which has been very useful for hyperbolic dynamical systems. The central point of their work was to prove the $C^r$-persistence of manifolds, foliations, or more generally laminations which are $r$-normally hyperbolic and plaque-expansive, for all $r \geq 1$.

We recall that a lamination is *$r$-normally hyperbolic*, if the dynamics preserves the lamination (each leaf is sent into a leaf) and if the normal space to the leaves splits into two $Tf$-invariant subspaces, that $Tf$ contracts (or expands) $r$-times more sharply than the tangent space to the leaves. Plaque expansiveness is a generalization [(1)] of expansiveness to the concept of laminations. The $C^r$-persistence of such a lamination means that for any $C^r$-perturbation of the dynamics, there exists a lamination, $C^r$-close to the first, which is preserved by the new dynamics, and such that the dynamics induced on the space of the leaves remains the same.

A direct application of this theory was the construction of an example of a robustly transitive diffeomorphism (every nearby diffeomorphism has a dense orbit) which is not Anosov. Then their work was used for example by C. Robinson [**28**] to prove the structural stability of $C^1$-diffeomorphisms that satisfy axiom A and the strong transversality condition.

Nowadays, this theory remains very useful in several mathematical fields such as generic dynamical systems, differentiable dynamics, foliations theory or Lie group theory.

Nevertheless, this theory is not optimal from several viewpoints.

There are laminations which are not normally hyperbolic but are persistent. For example, let $S$ be the 2-dimensional sphere and let $N$ be a compact manifold. Let $\mathcal{L}$ be the lamination structure on $N \times S$ whose leaves are the fibers of the canonical projection $N \times S \to S$. Let $f$ be the north-south dynamics on $S$. Let $F$ be the diffeomorphism on $N \times S$ equal to the product of the identity of $N$ with $f$. One

1. For instance a normally hyperbolic lamination, whose leaves are the fibers of a bundle, is plaque-expansive.

can easily show that for any diffeomorphism $F'$ close to $F$ in the $C^1$-topology, there exists a lamination structure $\mathcal{L}'$ on $N \times S$ which is preserved by $F'$ and is isomorphic to $\mathcal{L}$ by a map close to the identity. Here the lamination $\mathcal{L}$ is $C^1$-persistent, but is not 1-normally hyperbolic.

Furthermore, in his thesis, M. Shub [**30**] has shown that for a manifold $M$ and a $C^1$-endomorphism $f$, every compact set $K$ which is stable and expanded by $f$ is then structurally stable (any $C^1$-perturbation of $f$ preserves a compact subset, homeomorphic and $C^0$-close to $K$, such that via this homeomorphism the restriction of the dynamics to these compact sets are conjugate). By an endomorphism we mean a differentiable map, not necessarily bijective and possibly with some singularities.

Also, M. Viana [**35**] has used a persistent normally expanded lamination of (co-) dimension one to build a robustly non-uniformly expanding map. However, up to our knowledge, it has not been proved that a $r$-normally expanding and plaque-expansive lamination, by an endomorphism, is $C^r$-persistent. Yet, this result seems fundamental for the study of endomorphisms, and should be helpful in order to reduce the gap between the understanding of endomorphisms and of diffeomorphisms (structural stability, existence of new non-uniformly expanding maps, ...).

Finally, since Mañe's thesis [**19**], we know that a compact $C^1$-submanifold is (1)-normally hyperbolic if and only if it is $C^1$-persistent and uniformly locally maximal (*i.e.* there exists a neighborhood $U$ of the submanifold $N$ such that the maximal invariant subset in $U$ of any $C^1$-perturbation is a submanifold $C^1$-close to $N$). However, a uniform locally maximal submanifold $N$ can be persistent as a stratified space without being normally hyperbolic. For example, assume that a planar diffeomorphism has a hyperbolic fixed point $P$ with a one-dimensional stable manifold $X$. We suppose that $X$ without $\{P\}$ is contained in the repulsive basin of an expanding fixed point $R$ whose eigenvalues have same modulus. The set $\mathbb{S}$, equal to the union of $X$ and $\{R\}$, is homeomorphic to a circle. We may even define a stratification structure with $X$ and $\{R\}$ as strata. One easily shows that for any $C^1$-perturbation of the dynamics, there exists a hyperbolic fixed point $P'$ close to $P$ whose one-dimensional stable manifold $X'$ punctured by $P'$ belongs to the repulsive basin of a fixed point $R'$ close to $R$. In particular, there is a stratification $(X', \{R'\})$ on $\mathbb{S}' := X' \cup \{R'\}$ which is preserved by the perturbation of the dynamics, such that $X'$ is $C^1$-close to $X$, $\{R'\}$ is close to $\{R\}$ and $\mathbb{S}'$ is $C^0$-close to $\mathbb{S}$.

For these reasons, it seems useful and natural to wonder about the persistence of stratifications of normally expanded laminations, in the endomorphism context. As the concept of stratification of laminations is new, we are going to define all the above terms. Then we will give several applications of the main theorem of this work. At the end of this introduction, we will formulate a more transparent special case of the main theorem suitable for most of the presented applications.

### 0.2. Stratifications of normally expanded laminations

We recall that a lamination is a second-countable metric space $L$ locally modeled (via compatible charts) on the product of $\mathbb{R}^d$ with a locally compact space. The maximal set of compatible charts is denoted by $\mathcal{L}$.