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SYMMETRY TYPES OF HYPERELLIPTIC RIEMANN SURFACES

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Abstract. — A compact Riemann surface X is symmetric if it admits an antianalytic involution $\tau : X \to X$. Such an involution is called a real structure. Two real structures are isomorphic if they are conjugate in the full group $\mathrm{Aut}^{\pm}X$ of analytic and antianalytic automorphisms of X. In this memoir we classify up to isomorphism the real structures of all symmetric hyperelliptic Riemann surfaces of genus $g \geqslant 2$. The topological invariants of each isomorphism class are also computed. We give the list of groups which act as the full group of analytic and antianalytic automorphisms of such surfaces. Moreover, the complex algebraic curve associated to any such Riemann surface is described in terms of polynomial equations. We also find the explicit formula of a real structure in each isomorphism class.

***Résumé* (Types de symétrie des surfaces de Riemann hyperelliptiques)**

Une surface de Riemann compacte X est dite symétrique si elle admet une involution antiholomorphe $\tau : X \to X$. On appelle structure réelle une telle involution. Deux structures réelles sont isomorphes si elles sont conjuguées par le groupe complet $\mathrm{Aut}^{\pm}X$ des automorphismes holomorphes et anti-holomorphes de X. Dans ce mémoire, nous classifions à isomorphisme près les structures réelles de toutes les surfaces de Riemann hyperelliptiques de genre $g \geqslant 2$. Nous calculons aussi les invariants topologiques de chaque classe d'isomorphisme. Nous donnons la liste des groupes qui agissent comme le groupe des automorphismes holomorphes et anti-holomorphes d'une telle surface. De plus, nous décrivons la courbe algébrique complexe associée à une telle surface en terme d'équations polynomiales. Nous donnons enfin une formule explicite pour une structure réelle dans chaque classe d'isomorphisme.

CONTENTS

INTRODUCTION

Let X be a compact Riemann surface. A *real structure* of X is an antianalytic involution $\tau : X \to X$. We will also say that τ is a *symmetry* of X. The surface X is said to be symmetric if it admits some real structure τ. A *real form* of X is the conjugacy class of a real structure with respect to the group $\mathrm{Aut}^{\pm}X$ of all analytic and antianalytic automorphisms of X.

The origin of these names comes from the uniformization theorem of Koebe and Poincaré, since it implies that each compact Riemann surface is conformally equivalent to an irreducible smooth complex algebraic curve. Let $F_1, \ldots, F_m$ be a set of polynomials defining such a curve X. If each F_i turns out to have real coefficients then the complex conjugation determines an antianalytic involution τ on X. Thus, X is symmetric and τ is a real structure of X. A pair (X, τ) consisting of an irreducible smooth complex algebraic curve X and an antianalytic involution τ on it, is called a *real algebraic curve*. The complex curve X is said to be its *complexification*.

Most complex algebraic curves have no real form and others have more than one. For example, let X be the elliptic curve defined by

$$X = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{C}) : x_0 x_2^2 = x_1(x_1^2 + x_0^2)\}.$$

Clearly, the restriction τ to X of the complex conjugation on $\mathbb{P}^2$ is a real structure of X. Also $\varphi \circ \tau$ is a real structure of X, where φ is the birational automorphism of X given by

$$\varphi : [x_0 : x_1 : x_2] \longmapsto [x_0 : -x_1 : ix_2], \quad \text{where} \quad i = \sqrt{-1}.$$

It is easy to see that the fixed point set $\mathrm{Fix}(\tau)$ of τ has one connected component whilst $\mathrm{Fix}(\varphi \circ \tau)$ has two; hence τ and $\varphi \circ \tau$ are non-conjugate real structures of X and so (X, τ) and $(X, \varphi \circ \tau)$ are non-isomorphic real algebraic curves with the same complexification.

Along this memoir the terms "compact Riemann surface" and "complex algebraic curve" will be used indistinctly.

Let $k \geqslant 0$ be the number of connected components of $\mathrm{Fix}(\tau)$ and let ε be the *separability character* of τ defined as $\varepsilon = -1$ if $X - \mathrm{Fix}(\tau)$ is connected and $\varepsilon = 1$ otherwise. Note that we do not exclude the possibility of $\mathrm{Fix}(\tau)$ to be empty. The parameters k and ε classify τ topologically. Clearly a conjugate in $\mathrm{Aut}^{\pm}X$ of τ is also a symmetry with the same topological type; that is, k and ε just depend on the conjugacy class of τ. So we define the *species* $\mathrm{sp}(\tau)$ of the real form represented by τ to be the integer εk. The *symmetry type* of X is the (finite) set of species of all real forms of X.

The computation of the symmetry type of a hyperelliptic Riemann surface is a classic problem, posed by Felix Klein in 1893 and solved by himself in the case $|\mathrm{Aut}^{\pm} X| = 4$. Partial solutions have appeared since, as in the cases of low genus or special families of Riemann surfaces (see below). These solutions are immediate consequence of the results in this memoir since here we completely solve this problem. Namely, *we compute the symmetry types of all compact hyperelliptic Riemann surfaces of genus* $g \geqslant 2$.

We also obtain, for each $g \geqslant 2$, *the list of groups which act as the full group of analytic and antianalytic automorphisms of a genus g symmetric hyperelliptic Riemann surface.* This extends the results of Brandt and Stichtenoth in [**4**] and Bujalance, Gamboa and Gromadzki in [**15**].

The uniformization theorem makes the theory of Fuchsian groups a fruitfull technique to deal with compact Riemann surfaces. However, there is an increasing interest in describing them via defining equations. In this memoir *we compute explicit polynomial equations of each symmetric hyperelliptic Riemann surface. The formula of a representative of each real form is also given.*

The most elementary case for computing symmetry types is that of algebraic curves of genus zero. The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ admits exactly two real forms, namely those represented by the symmetries

$$\tau_1 : z \longmapsto \overline{z} \quad \text{and} \quad \tau_2 : z \longmapsto \frac{-1}{\overline{z}}.$$

The first fixes the real axis, which disconnects $\widehat{\mathbb{C}}$, while the second is fixed-point free. Therefore, the symmetry type of $\widehat{\mathbb{C}}$ is $\{1, 0\}$. The case of curves of genus one was completely solved by Alling [**1**]. Elliptic curves are tori, and each torus is isomorphic to the quotient $X_\gamma = \mathbb{C}/L_\gamma$ where $L_\gamma = \mathbb{Z} + \gamma\,\mathbb{Z}$ and $\gamma \in P = \{\gamma \in \mathbb{C} : |\gamma| \geqslant 1,\ |\mathrm{Re}(\gamma)| \leqslant 1/2\}$. The symmetric tori correspond to the points of P on the imaginary axis or on the boundary of P. For $\gamma = i$, the symmetry type of X_γ is $\{-1, 0, 2\}$. If $\mathrm{Re}(\gamma) = 0$ and $\mathrm{Im}(\gamma) > 1$ then the symmetry type of X_γ is $\{0, 0, 2, 2\}$, and the symmetry type of the other symmetric tori is $\{-1, -1\}$.

Among the pioneers in the study of real forms of a complex algebraic curve, Harnack [**32**], Weichold [**52**] and Klein [**35**] stand out. The first two determined the admissible values of the species of the real forms of a curve of genus g. As said above, Klein obtained the first result concerning the symmetry types of curves of genus $g \geqslant 2$. More precisely, he proved that the symmetry type of a hyperelliptic curve of genus $g \geqslant 2$ whose group of automorphisms has order 4 is one of the following: $\{-1, -1\}, \{-2, -2\}, \ldots, \{-g, -g\}$, $\{g + 1, g + 1\}$ or $\{0, 1\}$ if g is even, and $\{-1, -1\}, \{-2, -2\}, \ldots, \{-g, -g\}$, $\{g + 1, g + 1\}$, $\{0, 2\}$ or $\{0, 0\}$ if g is odd.

After these pioneer works, interest was lost in studying real forms of algebraic curves until the seventies, with the foundational research of Alling and Greenleaf [**2**], Earle [**24**] and Gross and Harris [**31**]. Moreover, the development of proper techniques of the real algebraic geometry, see for example the book of Bochnak, Coste and Roy [**3**], has propelled ahead this new field of research. Alling and Greenleaf studied systematically *Klein surfaces*, mainly compact ones, which may be seen as quotient spaces $X/\langle\tau\rangle$ where τ is a real structure of the compact Riemann surface X. With the obvious definition, conjugate real structures give rise to isomorphic Klein surfaces. They showed that the categories of compact Klein surfaces and real algebraic curves are equivalent. Earle introduced the moduli of compact Riemann surfaces with symmetries, while Gross and Harris showed, among other things, that the invariants k and ε of a symmetry are determined by the first homology group $H_1(X, \mathrm{Z}_2)$, and conversely. They also described the topology of hyperelliptic real algebraic curves.

Related to the problem of existence of symmetries in a Riemann surface, we mention here the work of Singerman [**48**]. He obtained conditions for a Riemann surface with large automorphism group to be symmetric. For example, he showed that all Riemann surfaces admitting automorphisms of order greater than $2g+2$ are symmetric. However, he also exhibited an example of a Riemann surface having Hurwitz automorphism group which is not symmetric. (A Riemann surface of genus $g \geqslant 2$ has Hurwitz automorphism group if it admits the maximum number $84(g-1)$ of automorphisms that a genus g Riemann surface may admit.)

In the same line of Klein's results quoted above, Bujalance and Singerman [**18**] calculated the 18 symmetry types of symmetric Riemann surfaces of genus 2. Since all such surfaces are hyperelliptic, these symmetry types appear naturally in this memoir. They showed, for example, that such a surface always admits a real form with non-zero species. They also characterized, in terms of the full group of automorphisms, the surfaces admitting a unique real form. Explicit polynomial equations for these surfaces and their real forms have been calculated by Cirre in [**21**], where the same description has also been done for the family of curves admitting the maximum number of real forms with non-zero species. More recently, Melekoğlu in [**39**] has calculated the symmetry types of curves of genus 3.

It is worthwhile mentioning other results in the same line. For example, Natanzon obtained in [**41**], [**42**] and [**43**] the symmetry types of those algebraic curves of genus g admitting a real form of species $g+1$ or $-g$. Using combinatorial methods, Bujalance and Costa in [**9**] also studied the symmetries of these curves. In [**12**] Bujalance, Costa and Gamboa calculated the symmetry types of the algebraic curves whose group of analytic automorphisms has prime order. This extends Klein's results quoted above. Bujalance and Costa [**10**] found the symmetry type $\{-2, 0\}$ of the famous Macbeath's curve of genus 7. It must be pointed out that the aid of the symbolic language CAYLEY has been very useful to compute finite generating sets and conjugacy classes of some groups, and also to decide the separability character of some symmetries. More recently, Broughton, Bujalance, Costa, Gamboa and Gromadzki in [**5**] and [**6**] obtained the symmetry types of those curves on which $\mathrm{PSL}(2,q)$ acts as a Hurwitz automorphism group, and of the Accola-Maclachlan and Kulkarni curves, respectively.

All these last results were obtained by the combinatorial methods to be explained below. By using purely algebraic arguments, Turbek [**51**] calculated the symmetry type of the so called Kulkarni curve. It should be remarked that only this last paper, [**6**] and [**21**] provide explicit formulae for the symmetries representing the real forms.

Sometimes it is helpful to know, before computing the symmetry type of an algebraic curve, the number of their real forms. To that end, some upper bounds have been obtained in the last twenty years. Natanzon [**44**], using topological methods, proved that an algebraic curve of genus g has at most $2(\sqrt{g}+1)$ real forms of nonzero species. He also showed that this bound is attained for infinitely many values of g, those of the form $g=(2^n-1)^2$. Later on, Bujalance, Gromadzki and Singerman [**17**] obtained a combinatorial proof of this result and proved that these are the only values of g for which the bound is sharp. This has been considerably improved recently by Bujalance, Gromadzki and Izquierdo [**16**]. If $g=1+2^{r-1}u$ with u odd, then every algebraic curve of genus g has at most 2^{r+1} real forms with nonzero species. In particular, it follows a striking corollary which was first proved by Gromadzki and Izquierdo [**30**]: each algebraic curve of even genus has at most 4 real forms with nonzero species. A bound for the number of real forms with zero species will appear in the paper [**8**] by Bujalance, Conder, Gamboa, Gromadzki and Izquierdo.

Also related with this subject we mention here the papers by Natanzon [**45**], Singerman [**50**] and Gromadzki [**28**], [**29**], where they get upper bounds for the sum of the number of connected components of the real structures of an algebraic curve. In particular the hyperelliptic case is treated.

Other results concerning topological properties of symmetries of Riemann surfaces have been obtained by Bujalance, Costa, Natanzon and Singerman in [**13**], Bujalance and Costa in [**11**] and Izquierdo and Singerman in [**34**].

Closely connected with the study of symmetries of hyperelliptic algebraic curves is that of the so called *pseudo-symmetries* due to Singerman [**49**]. Each symmetry τ of the hyperelliptic curve X induces a symmetry $\widehat{\tau}$ of the Riemann sphere $\widehat{\mathbb{C}}$. However the converse is not always true: some symmetries $\widehat{\tau}$ of $\widehat{\mathbb{C}}$ admit liftings $\tau: X \to X$ of order 4. They are called pseudo-symmetries and will appear in a natural way in our work.

Computational aspects in the theory of Riemann surfaces are an increasing subject of research. One of the main goals is to pass explicitly between defining equations, Fuchsian groups and period matrices. This is the classical uniformization problem. Among the recent results in this direction, we mention here the paper by Gianni, Seppälä, Silhol and Trager [**26**] where they have designed an algorithm to compute a