MÉMOIRES DE LA SMF 87

ALBANESE AND PICARD 1-MOTIVES

Luca Barbieri-Viale
Vasudevan Srinivas

Société Mathématique de France 2001
Publié avec le concours du Centre National de la Recherche Scientifique

L. Barbieri-Viale

Dipartimento di Metodi e Modelli Matematici, Università degli Studi di Roma "La Sapienza", Via A. Scarpa, 16, I-00161 Roma, Italia.

E-mail : barbieri@dmmm.uniroma1.it

Url : http://www.dmmm.uniroma1.it/

V. Srinivas

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India.

E-mail : srinivas@math.tifr.res.in

Url : http://www.math.tifr.res.in/

2000 *Mathematics Subject Classification.* — 14F42, 14C30, 32S35, 19E15.

Key words and phrases. — Hodge theory, motives, algebraic cycles, singularities.

This work has been done during periods of stay at several research institutions including University of Paris VII and Institut Henri Poincaré in Paris, T.I.F.R. in Mumbai, DI.MA in Genoa and I.C.T.P. in Trieste which we thank for the hospitality.

The second author wishes to acknowledge support from the Swarnajayanthi Fellowship of the D.S.T. and the IFCPAR Project "Geometry"(No. 1601-2), which made possible a visit to the University of Paris VII, concurrently with a visit of the first author to the Institut Henri Poincaré, and enabled useful discussions on this work with O. Gabber, L. Illusie, F. Oort and M. Raynaud. In particular, we thank Raynaud for providing us with the reference [**53**].

The first author is deeply grateful to the Italian C.N.R. and M.U.R.S.T., the European Union Science Plan "K-Theory and Algebraic Groups" and the Tata Institute of Fundamental Research for financial support.

ALBANESE AND PICARD 1-MOTIVES

Luca Barbieri-Viale, Vasudevan Srinivas

***Abstract.* —** Let X be an n-dimensional algebraic variety over a field of characteristic zero. We describe algebraically defined Deligne 1-motives $\mathrm{Alb}^+(X)$, $\mathrm{Alb}^-(X)$, $\mathrm{Pic}^+(X)$ and $\mathrm{Pic}^-(X)$ which generalize the classical Albanese and Picard varieties of a smooth projective variety. We compute Hodge, ℓ-adic and De Rham realizations proving Deligne's conjecture for H^{2n-1}, H_{2n-1}, H^1 and H_1.

We investigate functoriality, universality, homotopical invariance and invariance under formation of projective bundles. We compare our cohomological and homological 1-motives for normal schemes. For proper schemes, we obtain an Abel-Jacobi map from the (Levine-Weibel) Chow group of zero cycles to our cohomological Albanese 1-motive which is the universal regular homomorphism to semi-abelian varieties. By using this universal property we get "motivic" Gysin maps for projective local complete intersection morphisms.

***Résumé* (1-motifs d'Albanese et de Picard). —** Soit X une variété algébrique de dimension n sur un corps de caractéristique 0. Nous décrivons les 1-motifs de Deligne $\mathrm{Alb}^+(X)$, $\mathrm{Alb}^-(X)$, $\mathrm{Pic}^+(X)$ et $\mathrm{Pic}^-(X)$ définis algébriquement, qui généralisent les variétés d'Albanese et de Picard classiques d'une variété projective lisse. Nous calculons les réalisations de Hodge, ℓ-adique et de De Rham, montrant ainsi la conjecture de Deligne pour H^{2n-1}, H_{2n-1}, H^1 et H_1.

Nous étudions la fonctorialité, l'universalité, l'invariance par homotopie et l'invariance par formation de fibrés projectifs. Nous comparons nos 1-motifs homologiques et cohomologiques pour les schémas normaux. Pour des schémas propres, nous obtenons une application d'Abel-Jacobi du groupe de (Levine-Weibel) Chow des zéro-cycles vers notre 1-motif cohomologique d'Albanese, qui est l'homomorphisme universel régulier vers les variétés semi-abéliennes. En utilisant cette propriété universelle, nous obtenons des applications de Gysin « motiviques » pour les morphismes projectifs localement intersection complète.

CONTENTS

CHAPTER 0

INTRODUCTION

This work is motivated by Deligne's conjecture that 1-motives obtained from the mixed Hodge structure on the cohomology of an algebraic variety would be "algebraically defined" (see [**15**, § 10.4.1] and [**16**]). Deligne ([**15**, § 10.1.3]) observed that a torsion free mixed Hodge structure H (*i.e.*, such that $H_{\mathbb{Z}}$ is torsion-free), which is of Hodge type $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$, and such that $\mathrm{gr}^W_{-1}(H)$ is polarizable, yields *i)* a semi-abelian variety G, whose abelian quotient is the abelian variety given by $\mathrm{gr}^W_{-1}(H)$, together with *ii)* a homomorphism u from the lattice $L = \mathrm{gr}^W_0(H_{\mathbb{Z}})$ to the group $G(\mathbb{C})$, induced by the canonical map $H_{\mathbb{Z}} \to H_{\mathbb{C}}$.

Deligne called such a complex of group schemes $[L \xrightarrow{u} G]$ a 1-*motive over* $\mathbb{C}$, and showed that the category of 1-motives over $\mathbb{C}$ is equivalent to the category of torsion free mixed Hodge structures of the above type. Thus any such 1-motive $M = [L \xrightarrow{u} G]$ has a *Hodge realization* $T_{\mathrm{Hodge}}(M)$, *i.e.*, there is a unique (up to isomorphism) torsion-free mixed Hodge structure $T_{\mathrm{Hodge}}(M)$ such that M can be obtained from $T_{\mathrm{Hodge}}(M)$ as above. Deligne ([**15**, § 10.1.11]) also defined the ℓ-*adic* and *De Rham realizations* of a 1-motive M, denoted by $T_\ell(M)$ and $T_{\mathrm{DR}}(M)$, respectively (see Chapter 1 for more details).

0.1. The conjecture

Deligne's conjecture, in particular, is that if X is an n-dimensional algebraic variety over a field k of characteristic 0, then there are "algebraically defined" 1-motives, also defined over k, compatible with base change to extension fields, such that *i)* when $k = \mathbb{C}$, their Hodge realizations are respectively isomorphic to the mixed Hodge structures on

$$H^{2n-1}(X,\mathbb{Z}(n))/(\text{torsion}),\ \ H_1(X,\mathbb{Z})/(\text{torsion}),\ \ H^1(X,\mathbb{Z}(1)),\ \ H_{2n-1}(X,\mathbb{Z}(1-n))$$

ii) if k is algebraically closed, their ℓ-adic and De Rham realizations are naturally isomorphic to the corresponding ℓ-adic and De Rham (co)homology *iii)* the above identifications are compatible with other structures, like comparison isomorphisms, filtrations, Galois action, *etc*. Our goal is to prove these statements. Moreover, we obtain some geometric properties of our constructions.

We recall that the case $n = 1$, *i.e.*, when X is a curve, is already treated by Deligne ([**15**, § 10.3]), and the case when X smooth and proper corresponds to the well known transcendental descriptions of the Albanese and Picard varieties. Our construction of Albanese and Picard 1-motives generalizes these cases. In the general case, for $n \geqslant 1$, we propose the following dictionary:

Mixed Hodge Structure	1-*Motive*
$H^{2n-1}(X, \mathbb{Z}(n))$	$\mathrm{Alb}^+(X)$
$H_{2n-1}(X, \mathbb{Z}(1-n))$	$\mathrm{Pic}^-(X)$
$H^1(X, \mathbb{Z}(1))$	$\mathrm{Pic}^+(X)$
$H_1(X, \mathbb{Z})$	$\mathrm{Alb}^-(X)$

Here, $\mathrm{Alb}^+(X)$ is the "cohomological Albanese" 1-motive obtained from the mixed Hodge structure $H^{2n-1}(X, \mathbb{Z}(n))/(\text{torsion})$ and, dually, $\mathrm{Pic}^-(X)$ is the "homological Picard" 1-motive obtained from $H_{2n-1}(X, \mathbb{Z}(1-n))/(\text{torsion})$, *etc.* The 1-motive $\mathrm{Alb}^+(X)$ is the Cartier dual of $\mathrm{Pic}^-(X)$ and $\mathrm{Pic}^+(X)$ is the Cartier dual of $\mathrm{Alb}^-(X)$. In case X is singular, we have that $\mathrm{Alb}^+(X) \neq \mathrm{Alb}^-(X)$ in general, because of the possible failure of Poincaré duality. When $n = 1$, $\mathrm{Alb}^+(X)$ and $\mathrm{Pic}^+(X)$ (and dually, $\mathrm{Alb}^-(X)$ and $\mathrm{Pic}^-(X)$) coincide.

We recall that the geometric definition of the "cohomological Picard and homological Albanese" 1-motives of a smooth, but possibly non-proper scheme X, goes back to Serre's explicit construction of its Albanese variety, see [**48**]; in fact, Serre's Albanese variety was defined as the Cartier dual of the 1-motive

$$\mathrm{Pic}^+(X) \overset{\text{def}}{=} [\mathrm{Div}_Y^0(\overline{X}) \to \mathrm{Pic}^0(\overline{X})] \qquad (X \text{ smooth})$$

where $\overline{X}$ is a smooth compactification of X with boundary Y, $\mathrm{Div}_Y^0(\overline{X})$ is the free abelian group of divisors which are algebraically equivalent to zero and supported on Y, being mapped canonically to $\mathrm{Pic}^0(\overline{X})$. On the other hand, a geometric construction of Alb^+ or Pic^- for a smooth open variety is more difficult and it appears to be new as well.

Following the construction in [**31**], in the paper of Ramachandran [**39**] a geometric construction of $\mathrm{Pic}^+(X)$ and $\mathrm{Alb}^-(X)$ was proposed for varieties with a singular closed point obtained by collapsing a finite set of closed points in a smooth open variety; in a subsequent paper, see [**40**], he proposed, independently, definitions of Albanese and Picard motives corresponding to our $\mathrm{Pic}^+(X)$ and $\mathrm{Alb}^-(X)$. Ramachandran announced in [**41**] (*cf.* [**3**]) a proof of the algebraicity (up to isogeny) of certain 1-motives built out of $H^i(X, \mathbb{Q}(1))$ for $i \leqslant \dim X + 1$.

Apart from Ramachandran's work, a related paper by Carlson [**12**] on analogues of Abel's theorem for H^2 of some singular surfaces (see also [**22**]) and the recent paper [**3**] (see also the "Hodge 1-motives" considered in [**1**] and related papers [**7**] and [**9**]) we do not know any results on Deligne's conjecture (1972).

0.2. The results

Our definition of $\mathrm{Pic}^-(X)$ is roughly the following (see Sections 2.1 and 2.2 below for a more precise statement). Let X be any equidimensional algebraic variety over an algebraically closed field k of characteristic zero. Let $f:\widetilde{X}\to X$ be a resolution of singularities and let $\overline{X}$ be a smooth compactification of $\widetilde{X}$ with normal crossing boundary divisor Y.

Let S be the singular locus of X and let $\overline{S}$ be the closure of $f^{-1}(S)$ in $\overline{X}$. Then we let $\mathrm{Div}^0_{\overline{S}}(\overline{X},Y)$ be the group of divisors supported on $\overline{S}$ which are *i)* disjoint from Y (*i.e.*, are linear combinations of compact components of $f^{-1}(S)$), and *ii)* are algebraically equivalent to zero relative to Y. We let $\mathrm{Div}^0_{\overline{S}/S}(\overline{X},Y)$ be the subgroup of those divisors which have vanishing push-forward (as Weil divisors) along f.

We can show the existence of a group scheme $\mathrm{Pic}(\overline{X},Y)$ associated to isomorphism classes of pairs $(\mathcal{L},\varphi)$ such that $\mathcal{L}$ is a line bundle on $\overline{X}$ and $\varphi:\mathcal{L}\mid_Y\cong\mathcal{O}_Y$ is a trivialization on Y. The connected component of the identity $\mathrm{Pic}^0(\overline{X},Y)$ is a semi-abelian variety, which can be represented as an extension

$$0\to\frac{H^0(Y,\mathcal{O}_Y^*)}{\operatorname{im}H^0(\overline{X},\mathcal{O}_{\overline{X}}^*)}\to\mathrm{Pic}^0(\overline{X},Y)\to\ker^0(\mathrm{Pic}^0(\overline{X})\to\oplus_i\mathrm{Pic}^0(Y_i))\to 0$$

where $Y=\cup_iY_i$ is expressed as a union of (smooth) irreducible components. The mapping which takes a divisor D disjoint from Y to the class of the pair $(\mathcal{O}_{\overline{X}}(D),1)$ in $\mathrm{Pic}(\overline{X},Y)$ yields the "homological Picard" 1-motive

$$\mathrm{Pic}^-(X)\overset{\text{def}}{=}[\mathrm{Div}^0_{\overline{S}/S}(\overline{X},Y)\to\mathrm{Pic}^0(\overline{X},Y)].$$

The "cohomological Albanese" 1-motive $\mathrm{Alb}^+(X)$ is defined to be the Cartier dual of $\mathrm{Pic}^-(X)$ (see Section 3.1); a "concrete" description of it is also given when X is either smooth or proper.

The definition of $\mathrm{Pic}^+(X)$ is obtained by generalizing Serre's construction of the generalized Albanese variety to smooth simplicial schemes (see Sections 4.1 and 4.2 for the details). Let X be a variety over an algebraically closed field k of characteristic 0. Let $X_\bullet$ be a smooth proper hypercovering of X, and consider a proper smooth compactification $\overline{X}_\bullet$ with normal crossing boundary $Y_\bullet$ (we refer to [**15**, § 6.2] for the existence of such a hypercovering and compactification). Let $\mathrm{Div}_{Y_\bullet}(\overline{X}_\bullet)$ be the subgroup of divisors on $\overline{X}_0$ supported on Y_0 with zero pull-back on $\overline{X}_1$, *i.e.*, by definition

$$\mathrm{Div}_{Y_\bullet}(\overline{X}_\bullet)\overset{\text{def}}{=}\ker(\mathrm{Div}_{Y_0}(\overline{X}_0)\xrightarrow{d_0^*-d_1^*}\mathrm{Div}_{Y_1}(\overline{X}_1)).$$

We consider the simplicial Picard functor

$$T \longmapsto \mathbf{Pic}(T \times \overline{X}_{\bullet}) \stackrel{\text{def}}{=} \mathbb{H}^1(T \times \overline{X}_{\bullet}, \mathcal{O}^*_{T\times\overline{X}_{\bullet}})$$

and we show that the associated sheaf $\mathbf{Pic}_{\overline{X}_{\bullet}/k}$ (with respect to the *fpqc*-topology) is representable by a group scheme locally of finite type over k, whose connected component of the identity $\mathbf{Pic}^0_{\overline{X}_{\bullet}/k}$ over $k = \overline{k}$ is an extension of the abelian scheme $\ker^0(\mathrm{Pic}^0_{\overline{X}_0/k} \to \mathrm{Pic}^0_{\overline{X}_1/k})$ by the torus given by

$$\frac{\ker(H^0(\overline{X}_1, \mathcal{O}^*_{\overline{X}_1}) \to H^0(\overline{X}_2, \mathcal{O}^*_{\overline{X}_2}))}{\mathrm{im}(H^0(\overline{X}_0, \mathcal{O}^*_{\overline{X}_0}) \to H^0(\overline{X}_1, \mathcal{O}^*_{\overline{X}_1}))}.$$

Let $\mathrm{Div}^0_{Y_{\bullet}}(\overline{X}_{\bullet})$ denote the subgroup of those divisors which are mapped to $\mathbf{Pic}^0_{\overline{X}_{\bullet}/k}(k)$ under the canonical mapping. We then define the "cohomological Picard" 1-motive of the variety X as

$$\mathrm{Pic}^+(X) \stackrel{\text{def}}{=} [\mathrm{Div}^0_{Y_{\bullet}}(\overline{X}_{\bullet}) \to \mathbf{Pic}^0(\overline{X}_{\bullet})].$$

The "homological Albanese" 1-motive $\mathrm{Alb}^-(X)$ is defined to be the Cartier dual of $\mathrm{Pic}^+(X)$ (see Section 5.1).

We show that $\mathrm{Pic}^-(X)$, $\mathrm{Alb}^+(X)$, $\mathrm{Pic}^+(X)$ and $\mathrm{Alb}^-(X)$ do have the appropriate Hodge, De Rham and ℓ-adic realizations (in Sections 2.4–2.6, 3.3, 4.3–4.5 and 5.3 respectively). We mostly deal with the geometric case, *i.e.*, we consider varieties X over an algebraically closed field k; the case when k is not algebraically closed is considered in Chapter 7.

We show as well that our definitions are functorial and independent of choices of resolutions or compactifications (*e.g.*, see Section 2.3) and depend only on the semi-normalization of the given variety (see Section 6.1). We remark (in Section 6.2) that Alb^+ can be contravariant functorial only for morphisms between varieties of the same dimension, and similarly Pic^- is covariant functorial for such maps. We then show the homotopical invariance of Pic^+ (and hence dually of Alb^-), and that Pic^- and Pic^+ (and dually, the corresponding Albanese 1-motives) are invariant under formation of projective bundles (see Section 6.3).

For proper X, we remark that our "cohomological" Albanese 1-motive $\mathrm{Alb}^+(X)$ is a quotient of Serre's Albanese of the regular locus X_{reg}, *i.e.*, we have an extension

$$0 \to T(S) \to \mathrm{Alb}^-(X_{\mathrm{reg}}) \to \mathrm{Alb}^+(X) \to 0$$

where $T(S)$ is a torus whose character group is a sublattice of the lattice of Weil divisors which are supported on the singular locus S. Thus, if X is also irreducible and normal, then $T(S) = 0$, and further, any non-zero Cartier divisor supported on the exceptional locus of a resolution is not numerically equivalent to zero; therefore, $\mathrm{Alb}^-(X_{\mathrm{reg}}) = \mathrm{Alb}^+(X)$ is an abelian variety which is isomorphic to the Albanese variety $\mathrm{Alb}(\widetilde{X})$ of any resolution of singularities $\widetilde{X}$ of X. In general, $\mathrm{Alb}^-(X_{\mathrm{reg}})$ is a torus bundle over $\mathrm{Alb}(\widetilde{X})$ whose pull-back to X_{reg} (under a suitable Albanese