

MÉMOIRES DE LA SMF 88

THE WAVE DIFFRACTED
BY A WEDGE WITH
MIXED BOUNDARY CONDITIONS

Olivier Lafitte

Société Mathématique de France
Publié avec le concours du Centre National de la Recherche Scientifique

O. Lafitte

Commissariat à l'Énergie Atomique-DAM., DCSA/MLS,
Centre d'Études de Bruyères-le-Chatel, 91680 Bruyères-le-Chatel.

Massachusetts Institute of Technology, Math. Department, Cambridge, MA 02 139,
USA.

DM2S/DIR., Centre d'études de Saclay, 91191 Gif-sur-Yvette Cedex.

2000 Mathematics Subject Classification. — Primary 35L05, 35A21; Secondary 35Q60, 78A25.

Key words and phrases. — Diffraction, electromagnetic waves, wedge, impedance.

This work has been supported in part by the contract 93-1187/A000 of the Délégation Générale pour l'Armement (D.R.E.T).

The author completed this work while he was at the C.M.L.A., ENS Cachan.

He revised this work while in CMAT, École Polytechnique, 91128 Palaiseau Cedex.

The author wants to thank R.B. Melrose and the Massachusetts Institute of Technology for the year he spent in 1993-1994, when he started the present study.

À Christine, Marine et Geoffrey

THE WAVE DIFFRACTED BY A WEDGE WITH MIXED BOUNDARY CONDITIONS

Olivier Lafitte

Abstract. — We study the diffraction of a conormal wave by a curved wedge in $\mathbb{R}^2$, each face $+$ or $-$ of the wedge being characterized by a mixed boundary condition of impedance type $\partial_n u + z^\pm(x)\partial_t u = 0$. We reduce the problem to a system on the two traces of the diffracted wave on each face of the wedge. The principal matricial symbol of this system is the matrix of the “straightened” system obtained with the tangent diedra and with the boundary condition $\partial_n u + z^\pm(0)\partial_t u = 0$.

Résumé (L’onde diffusée par une arête avec conditions au bord mixtes)

Nous étudions la diffusion d’une onde conormale analytique par une arête (ou un dièdre) à faces courbes, muni de conditions de type impédance sur chaque face, de la forme $\partial_n u + z^\pm(x)\partial_t u = 0$. Nous ramenons ce problème à l’étude du système sur les traces et les dérivées normales sur chaque face. Ce système a pour terme principal le système obtenu en remplaçant chaque face par la face tangente et les conditions au bord par $\partial_n u + z^\pm(0)\partial_t u = 0$ et nous montrons que le système principal est inversible.

CONTENTS

0. Introduction	1
0.1. Statement of the problem	1
0.2. Rigorous asymptotic behavior of the solution	5
1. Existence, uniqueness, and regularity of the solution	13
1.1. Uniqueness and existence of a solution	13
1.2. The application to the system of Maxwell equations	15
1.3. Estimates on the boundary $\partial\mathcal{O} \times [0, T]$	18
2. Diffraction by a straight wedge	25
2.1. Existence of the solution of an elliptic problem	25
2.2. The mirror operator for a straight wedge	27
2.3. Identification of $H^1(\mathbb{R})$	30
2.4. Expression of the boundary operators using the Mellin transform	32
2.5. The fundamental analytic system for the traces	35
2.6. The invertibility in $L \times L$ of the analytic system	36
3. Microlocal propagation of singularities for the mixed problem	43
4. Reduced system on the boundary	63
4.1. The fundamental identity	64
4.2. Expression of the Neumann to Neumann operators	66
4.3. The Dirichlet to Neumann operators	67
4.4. Regularity of the operators $R_{+,0}^{\pm,k}$ and $R_{-,0}^{\pm}$	73
4.5. Asymptotic expansions of the operators $K_{\pm}^{+,k}$ and $R_{\pm,0}^{+,k}$	78
4.6. System of equations and microlocalisation of the sources	89
4.7. Inversion of the principal symbol	98

5. Calculus of the diffraction coefficient	103
5.1. Diffraction by a curved antenna	103
5.2. Calculus of the total diffraction coefficient	109
5.3. Calculus of u_e	112
5.4. Proof of Theorem 1	116
6. Appendix	121
6.1. Properties of the Helmholtz equations around corners	121
6.2. Calculus of the distributions on the boundary	127
6.3. Asymptotic expansions of the DTN operators (from Gérard and Lebeau)	145
Bibliography	165

CHAPTER 0

INTRODUCTION

0.1. Statement of the problem

Let $F = \{(x, y) \in \mathbb{R}^2, x \geq 0, b(x) \leq y \leq a(x)\}$ be a wedge in $\mathbb{R}^2$, the curves $y = a(x)$ and $y = b(x)$ are called in this paper the faces of the wedge. We assume $a(0) = b(0) = 0$, $a(x) > 0 > b(x)$ for $x > 0$:

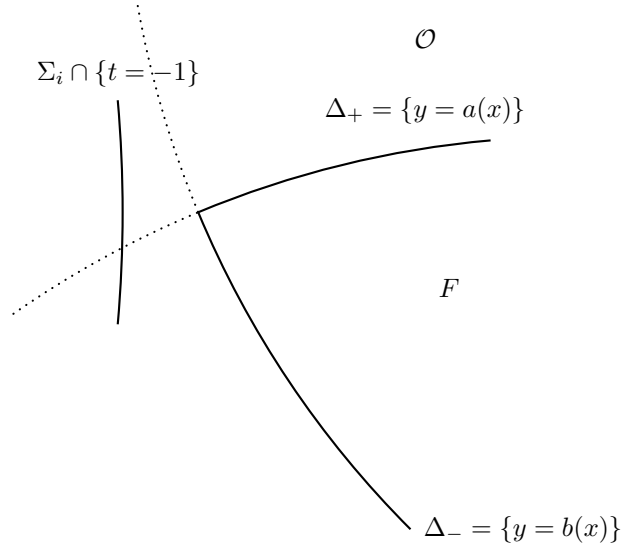


FIGURE 0.1. Wedge F , incident wave front set Σ_i

The space domain is $\mathcal{O} = \mathbb{R}^2 - F$. We denote the faces of the wedge by Δ_+ and Δ_- . With the notation $\tilde{\Delta}_+ = \{(x, y, t), y = a(x), t \in \mathbb{R}, x \in \mathbb{R}\}$ and $\tilde{\Delta}_- = \{(x, y, t), y = b(x), t \in \mathbb{R}, x \in \mathbb{R}\}$, we verify that $\Delta_{\pm} = \tilde{\Delta}_{\pm} \cap \{x > 0\}$. The functions

a and b are assumed to be analytic functions on $\mathbb{R}$. We define the *exterior domain* $\Omega = \mathcal{O} \times \mathbb{R}_t$. It will be convenient to consider that $\partial\mathcal{O} = \partial\mathcal{O}_+ \cup \partial\mathcal{O}_- \cup \{(0, 0)\}$. In all the sequel, p will denote the projection from $\mathbb{R}^2 \times \mathbb{R}_t$ to $\mathbb{R}^2$.

The problem of diffraction by a wedge with Dirichlet or Neumann boundary conditions has been studied by other authors before (starting with Poincaré [34], [35] and Sommerfeld [40], then Garnir [19], Bernard [3], [5], [7], [8], Kaminetzki-Keller [23], Bouche-Molinet [9], [30], Cessenat [12], Assous-Ciarlet [1]). However, these authors considered a wedge with straight faces or a wedge whose faces are circular arcs (Bernard [5]). The generalization to a curved wedge with analytic faces was done for a Dirichlet boundary condition by Gérard and Lebeau [20]. Other authors studied related problems: Kondrat'ev [24] considered more general boundary conditions and a cone, as well as Eskin [17, 18] or Bernard [7]. Grisvard [21], Azaiez-Dauge [2], Assous-Ciarlet-Sonnendruker [1] studied elliptic problems outside polyhedra.

The results of Gérard and Lebeau were used by Burq [11] to obtain a control result with open sets with corners. A generalization of the propagation result to a wedge in $\mathbb{R}^d$ is due to Lebeau [29].

We generalize in this paper the results of [20] for more general boundary conditions. We assume that each face of the wedge is characterized by an impedance boundary condition, that we describe below by equation (6).

Let us consider an incident wave $u_i(x, y, t) \in H_{\text{loc}}^1(\mathbb{R}^2 \times \mathbb{R}_t)$, solution of the wave equation $(\Delta - \partial_{t^2})u_i(x, y, t) = 0$. We assume that u_i is *conormal analytic* to a surface Σ_i such that $\Sigma_i \cap \{t < 0\} \subset \Omega$, $\Sigma_i \cap \{t = 0\} \cap \partial\Omega = \{(0, 0, 0)\}$ and $u_i(x, y, -\delta)$ is supported on the side of $p(\Sigma_i \cap \{t = -\delta\}) \subset \mathbb{R}^2$ which does not contain F (see Figure 0.1). The wave u_i is the generalization of a plane wave⁽¹⁾. This wave can be written, in a neighborhood of $t = x = y = 0$

$$(1) \quad u_i(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega(t - \theta_i(x, y))} \sigma_i(x, y, \omega) d\omega + a(x, y, t)$$

where the function a is analytic in the neighborhood of $(0, 0, 0)$ and $\theta_i(0, 0) = 0$, $\nabla\theta_i(0, 0) = (1, 0)$. The symbol σ_i is analytic and satisfies

$$(2) \quad \sup_{s \geq 0} \int_{-\infty}^{+\infty} (1 + |\tau|^2) |\sigma_i(x, y, \tau - is)|^2 d\tau < +\infty.$$

Let us define the impedance boundary conditions. For this purpose, we define, when they exist, the two traces ∂_+ and ∂_- , which are the normal derivatives on each face of the wedge (unlike in [20], where the normalization coefficient was not present):

$$(3) \quad \begin{cases} \partial_+ f(x) = (1 + (a'(x))^2)^{-1/2} (\partial_y f - a'(x) \partial_x f)|_{y-a(x)=0^+} \\ \partial_- f(x) = (1 + (b'(x))^2)^{-1/2} (b'(x) \partial_x f - \partial_y f)|_{y-b(x)=0^-} \end{cases}$$

⁽¹⁾For example, when $\sigma_i(x, y, \omega) = (1 + |\omega|)^{-3}$ and $\theta_i(x, y) = x$, u_i is (up to a regularization) the inverse Fourier transform of what is called a plane wave propagating in the x direction.