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THE WAVE DIFFRACTED BY A WEDGE WITH MIXED BOUNDARY CONDITIONS

Olivier Lafitte

Société Mathématique de France
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O. Lafitte
Commissariat à l'Énergie Atomique-DAM., DCSA/MLS,
Centre d'Études de Bruyères-le-Chatel, 91680 Bruyères-le-Chatel.
Massachusetts Institute of Technology, Math. Department, Cambridge, MA 02 139, USA.
DM2S/DIR., Centre d'études de Saclay, 91191 Gif-sur-Yvette Cedex.

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À Christine, Marine et Geoffrey

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Abstract. — We study the diffraction of a conormal wave by a curved wedge in $\mathbb{R}^2$, each face $+$ or $-$ of the wedge being characterized by a mixed boundary condition of impedance type $\partial_n u + z^{\pm}(x)\partial_t u = 0$. We reduce the problem to a system on the two traces of the diffracted wave on each face of the wedge. The principal matricial symbol of this system is the matrix of the "straightened" system obtained with the tangent diedra and with the boundary condition $\partial_n u + z^{\pm}(0)\partial_t u = 0$.

Résumé **(L'onde diffusée par une arête avec conditions au bord mixtes)**

Nous étudions la diffusion d'une onde conormale analytique par une arête (ou un dièdre) à faces courbes, muni de conditions de type impédance sur chaque face, de la forme $\partial_n u + z^{\pm}(x)\partial_t u = 0$. Nous ramenons ce problème à l'étude du système sur les traces et les dérivées normales sur chaque face. Ce système a pour terme principal le système obtenu en remplaçant chaque face par la face tangente et les conditions au bord par $\partial_n u + z^{\pm}(0)\partial_t u = 0$ et nous montrons que le système principal est inversible.

CONTENTS

CHAPTER 0

INTRODUCTION

0.1. Statement of the problem

Let $F = \{(x, y) \in \mathbb{R}^2,\ x \geqslant 0, b(x) \leqslant y \leqslant a(x)\}$ be a wedge in $\mathbb{R}^2$, the curves $y = a(x)$ and $y = b(x)$ are called in this paper the faces of the wedge. We assume $a(0) = b(0) = 0$, $a(x) > 0 > b(x)$ for $x > 0$:

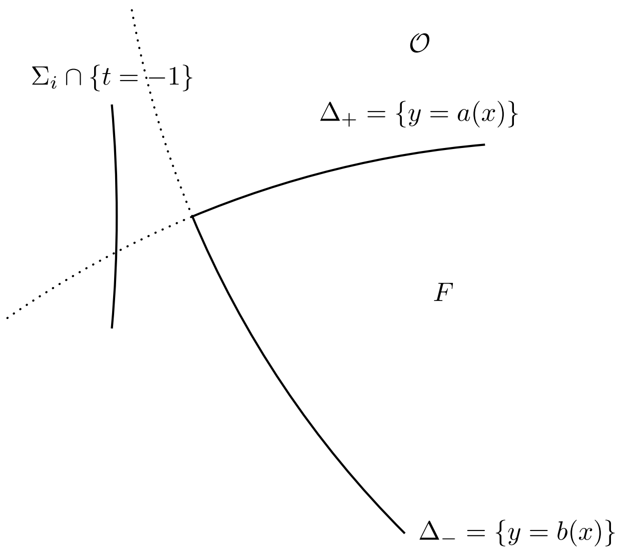

FIGURE 0.1. Wedge F, incident wave front set Σ_i

The space domain is $\mathcal{O} = \mathbb{R}^2 - F$. We denote the faces of the wedge by Δ_+ and Δ_-. With the notation $\widetilde{\Delta}_+ = \{(x, y, t),\ y = a(x),\ t \in \mathbb{R}, x \in \mathbb{R}\}$ and $\widetilde{\Delta}_- = \{(x, y, t),\ y = b(x),\ t \in \mathbb{R}, x \in \mathbb{R}\}$, we verify that $\Delta_\pm = \widetilde{\Delta}_\pm \cap \{x > 0\}$. The functions

a and b are assumed to be analytic functions on $\mathbb{R}$. We define the *exterior domain* $\Omega = \mathcal{O} \times \mathbb{R}_t$. It will be convenient to consider that $\partial\mathcal{O} = \partial\mathcal{O}_+ \cup \partial\mathcal{O}_- \cup \{(0,0)\}$. In all the sequel, p will denote the projection from $\mathbb{R}^2 \times \mathbb{R}_t$ to $\mathbb{R}^2$.

The problem of diffraction by a wedge with Dirichlet or Neumann boundary conditions has been studied by other authors before (starting with Poincaré [**34**], [**35**] and Sommerfeld [**40**], then Garnir [**19**], Bernard [**3**], [**5**], [**7**], [**8**], Kaminetzki-Keller [**23**], Bouche-Molinet [**9**], [**30**], Cessenat [**12**], Assous-Ciarlet [**1**]). However, these authors considered a wedge with straight faces or a wedge whose faces are circular arcs (Bernard [**5**]). The generalization to a curved wedge with analytic faces was done for a Dirichlet boundary condition by Gérard and Lebeau [**20**]. Other authors studied related problems: Kondrat'ev [**24**] considered more general boundary conditions and a cone, as well as Eskin [**17, 18**] or Bernard [**7**]. Grisvard [**21**], Azaiez-Dauge [**2**], Assous-Ciarlet-Sonnendrucker [**1**] studied elliptic problems outside polyhedra.

The results of Gérard and Lebeau were used by Burq [**11**] to obtain a control result with open sets with corners. A generalization of the propagation result to a wedge in $\mathbb{R}^d$ is due to Lebeau [**29**].

We generalize in this paper the results of [**20**] for more general boundary conditions. We assume that each face of the wedge is characterized by an impedance boundary condition, that we describe below by equation (6).

Let us consider an incident wave $u_i(x,y,t) \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}_t)$, solution of the wave equation $(\Delta - \partial^2_{t^2})u_i(x,y,t) = 0$. We assume that u_i is *conormal analytic* to a surface Σ_i such that $\Sigma_i \cap \{t < 0\} \subset \Omega$, $\Sigma_i \cap \{t = 0\} \cap \partial\Omega = \{(0,0,0)\}$ and $u_i(x,y,-\delta)$ is supported on the side of $p(\Sigma_i \cap \{t = -\delta\}) \subset \mathbb{R}^2$ which does not contain F (see Figure 0.1). The wave u_i is the generalization of a plane wave[(1)]. This wave can be written, in a neighborhood of $t = x = y = 0$

$$u_i(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(t-\theta_i(x,y))}\sigma_i(x,y,\omega)\,\mathrm{d}\omega + a(x,y,t) \tag{1}$$

where the function a is analytic in the neighborhood of $(0,0,0)$ and $\theta_i(0,0) = 0$, $\nabla\theta_i(0,0) = (1,0)$. The symbol σ_i is analytic and satisfies

$$\sup_{s \geqslant 0}\int_{-\infty}^{+\infty}(1+|\tau|^2)|\sigma_i(x,y,\tau - is)|^2\,\mathrm{d}\tau < +\infty. \tag{2}$$

Let us define the impedance boundary conditions. For this purpose, we define, when they exist, the two traces ∂_+ and ∂_-, which are the normal derivatives on each face of the wedge (unlike in [**20**], where the normalization coefficient was not present):

$$\begin{cases} \partial_+ f(x) = (1+(a'(x))^2)^{-1/2}(\partial_y f - a'(x)\partial_x f)|_{y-a(x)=0^+} \\ \partial_- f(x) = (1+(b'(x))^2)^{-1/2}(b'(x)\partial_x f - \partial_y f)|_{y-b(x)=0^-}. \end{cases} \tag{3}$$

(1) For example, when $\sigma_i(x,y,\omega) = (1+|\omega|)^{-3}$ and $\theta_i(x,y) = x$, u_i is (up to a regularization) the inverse Fourier transform of what is called a plane wave propagating in the x direction.