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GLOBAL SOLUTIONS FOR SMALL NONLINEAR LONG RANGE PERTURBATIONS OF TWO DIMENSIONAL SCHRÖDINGER EQUATIONS

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Abstract. — Let Q_1, Q_2 be two quadratic forms, and u a local solution of the two dimensional Schrödinger equation $(i\partial_t + \Delta)u = Q_1(u, \nabla_x u) + Q_2(\overline{u}, \nabla_x \overline{u})$. We prove that if Q_1 and Q_2 do depend on the derivatives of u, and if the Cauchy datum is small enough and decaying enough at infinity, the solution exists for all times. The difficulty of the problem originates in the fact that the nonlinear perturbation is a long range one: by this, we mean that it can be written as the product of (a derivative of) u and of a potential whose L^∞ space-norm is not time integrable at infinity.

Résumé **(Solutions globales pour des perturbations nonlinéaires à longue portée de l'équation de Schrödinger en dimension 2)**

Soient Q_1, Q_2 deux formes quadratiques et u solution locale de l'équation de Schrödinger en dimension 2 d'espace $(i\partial_t+\Delta)u = Q_1(u, \nabla_x u)+Q_2(\overline{u}, \nabla_x \overline{u})$. Nous prouvons que si Q_1 et Q_2 dépendent effectivement des dérivées de u, et si la donnée de Cauchy est assez petite et assez décroissante à l'infini, la solution existe globalement en temps. La difficulté du problème réside dans le fait que la perturbation nonlinéaire est à longue portée, en ce sens qu'elle s'écrit comme un produit (d'une dérivée) de u par un potentiel dont la norme L^∞ en espace n'est pas intégrable lorsque $t \to +\infty$.

CONTENTS

INTRODUCTION

This paper is devoted to the proof of existence of global solutions for a nonlinear Schrödinger equation in two space dimension with small Cauchy data. Consider the equation

$$
\begin{aligned}
(i\partial_t + \Delta)u &= F(u, \nabla_x u, \overline{u}, \nabla_x \overline{u}) \\
u|_{t=0} &= \varepsilon u_0
\end{aligned}
$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, F is a polynomial vanishing at least at order 2 at 0 and $\varepsilon > 0$.

The problem of *local existence* for the above equation with a general nonlinearity and for *small Cauchy data* (i.e. small ε) in a convenient Sobolev space has been solved by Kenig, Ponce and Vega [**19**]. Hayashi and Ozawa [**18**] obtained local existence in one space dimension for *large Cauchy data*. The case of any space dimension was treated by Chihara [**4**]. More recently, Kenig, Ponce and Vega [**20**] proved the similar result for a generalized Schrödinger equation, i.e. an equation in which Δ is replaced by a more general operator.

We are interested in this paper in *global solutions* for *small* enough ε. When the space dimension is larger or equal to 3, and F vanishes *at least at order* 3 at 0, Chihara [**5**], [**6**] proved that there is a global solution if the data are small enough in a (weighted) Sobolev space. He also proved the same result in two space dimensions under a convenient restriction on the cubic part of the nonlinearity.

For *quadratic nonlinearities*, and space dimension larger or equal to three, global existence for small data has been obtained under convenient assumptions on the nonlinearity. The most recent results are due to Hayashi and Hirata [**10**], Hayashi and Kato [**11**], Hayashi, Miao and Naumkin [**12**]. We refer the reader to the introduction of [**15**] for a detailed discussion of these results as well as further references.

The results we have mentioned so far could be qualified of “short range” type ones. By this, we mean the following: the nonlinearity F can be written as a sum of products of a nonlinear potential $V(u, \nabla_x u, \overline{u}, \nabla_x \overline{u})$ times u or $\nabla_x u$ or $\overline{u}$ or $\nabla_x \overline{u}$.

Denote by $k \geqslant 1$ the order of vanishing of V at the origin. Since linear solutions of the Schrödinger operator decay in L^∞ like $t^{-d/2}$ when $t \to +\infty$, we see that V computed on such a solution decays like $t^{-kd/2}$ when $t \to +\infty$. We say that F is a short range perturbation of the linear Schrödinger equation if this quantity is integrable when $t \to +\infty$ i.e. if $kd/2 > 1$. All the results we have indicated above fall into this category.

We are interested in this paper in the long range case, more precisely in the limiting case $kd/2 = 1$. There are only two such possibilities: either the space dimension d is 1 and F is cubic, or $d = 2$ and F is quadratic. The former case has been solved in general by Hayashi and Naumkin [**14**]: they found a sufficient condition on the cubic nonlinearity under which solutions are global for small enough Cauchy data in a weighted Sobolev space. Their method relies on the use of the smoothing property of Doi [**9**]. They could also in [**17**] reduce a particular quadratic nonlinearity to a cubic one, thus obtaining global existence in this case as well.

The case of *quadratic nonlinearities in two space dimensions* is studied by Cohn [**7**] for a very specific nonlinearity, and by Hayashi and Naumkin [**15**], [**16**] in the special case of *real analytic Cauchy data*. Such an assumption allows one to avoid the difficulty of the loss of one derivative in the right hand side of the equation.

Our aim in this paper is to study this quadratic two dimensional problem when the Cauchy datum lies in a weighted Sobolev space. We are thus obliged to cope with the problem of recovering the derivative lost in the right hand side. We state our main theorem of global existence in the first chapter, together with precise assumption on the quadratic nonlinearity we consider. Let us just describe here our general strategy in the special case

$$(D_t + D_x^2)u = u(D_{x_1}u)$$

where $D_t = \frac{1}{i}\partial_t$, $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 1, 2$, and where the datum is given at $t = 1$, the solution being looked for on $\{t \geqslant 1\}$. We first take new coordinates $T = t$, $X = x/t$ and look for u in terms of a new unknown $w(T, X)$ through $u(t, x) = \frac{1}{t}e^{ix^2/4t}w(t, x/t)$. We get for w an equation of form

$$\left(D_T + \frac{D_X^2}{T^2}\right)w = \frac{1}{T}e^{i\theta(T,X)}w\left(\frac{D_{X_1}}{T} + \frac{X_1}{2}\right)w \tag{0.1}$$

where $\theta = TX^2/4$. Let us take a simplified model forgetting the $X_1/2$ term above:

$$\left(D_T + \frac{D_X^2}{T^2}\right)w = \frac{1}{T}e^{i\theta(T,X)}w\left(\frac{D_{X_1}}{T}w\right). \tag{0.2}$$

Remark that D_{X_j} is the translation in the new coordinates of the operator $tD_{x_j} - x_j/2$, which is of constant use in the study of global problems for nonlinear Schrödinger equations with small Cauchy data. Consequently, smoothness relatively to D_X will play an essential role. The form of the right hand side of (0.2) shows immediately what are the difficulties we will encounter. First of all, we have a loss of a D_X/T derivative in the nonlinearity. To remove this problem, we shall use the Kato local smoothing property in the version due to Kenig, Ponce and Vega [**20**], adapted to our long time framework. Secondly, the right hand side of the equation contains the oscillating factor $e^{i\theta}$, which cannot have any D_X-smoothness uniformly as time $T \to +\infty$. To treat these oscillating contributions, we introduce spaces of the following type

$$\{v \in L^2; (D_X/\sqrt{T})^s (D_X/T)^{s'} v \in L^2\} \tag{0.3}$$

where s and s' are integers. The smoothness relatively to D_X/T corresponds to what is gained by the local smoothing property – and to what is lost in the nonlinearity $w\frac{D_{X_1}}{T}w$. The smoothness relatively to $D_X/\sqrt{T}$ should be understood as a weak version of smoothness relatively to $tD_x - x/2$ for $u(t,x)$. This type of derivative is natural for the problem because of the form of $\theta(T,X) = (\sqrt{T}X)^2/4$. To study products of elements in (0.3), we will need to have s and s' large enough. There will be no problem to ensure that for s', but as $\frac{D_X}{\sqrt{T}}e^{i\theta} = \frac{\sqrt{T}X}{2}e^{i\theta}$ we cannot expect the right hand side of (0.2) to be in a space of type (0.3) with a positive s. Consequently, instead of trying to find directly w in a space of type (0.3), we shall look for w as an expansion $w = v + V_1(v)e^{i\theta}$ where v and $V_1(v)$ will be essentially in a space (0.3) with large enough s, s', and where $V_1(v)$ will moreover decay like $(\sqrt{T}|X|)^{-2}$ when $\sqrt{T}|X| \to +\infty$. When plugging such an expression in $e^{i\theta(T,X)}w(\frac{D_{X_1}}{T}w)$, one gets a first contribution of form $e^{i\theta(T,X)}v(\frac{D_{X_1}}{T}v)$, and remainders decaying like $\langle\sqrt{T}X\rangle^{-2}$. One will choose $V_1(v)$ as a function of v such that $(D_T + \frac{D_X^2}{T^2})(V_1 e^{i\theta})$ equals $\frac{1}{T}v(\frac{D_{X_1}}{T}v)e^{i\theta}$ modulo remainders. This is possible because θ is a non characteristic phase for the operator $D_T + \frac{D_X^2}{T^2}$. In that way, one gets an equation

$$\left(D_T + \frac{D_X^2}{T^2}\right)v = \frac{1}{T}R \tag{0.4}$$

where R will be a combination of terms $e^{im\theta}$ with coefficients decaying at least like $\langle\sqrt{T}X\rangle^{-2}$. Since $\langle\sqrt{T}X\rangle^{-2}e^{im\theta}$ has some smoothness relatively to $D_X/\sqrt{T}$ uniformly in T (actually, this expression accepts two $D_X/\sqrt{T}$ derivatives), this shows that we have gained some smoothness in comparison with the right hand side of (0.2). Actually one has to repeat such a method once again, to reduce the equation to (0.4) with a right hand side $R = R(v, \frac{D_X}{T}v)$ with values in a space of type (0.3) with $s \sim 4$. This last equation can then be solved globally using the local smoothing property as mentioned before.

Our paper is organized as follows. In the first chapter, we state our main theorem and perform first reductions. The second chapter is devoted to the proof of the local smoothing inequality that will be essential in the rest of the article. We make use of Littlewood-Paley decompositions to define convenient Sobolev spaces, and to prove the smoothing inequality as a consequence of propagation of singularities. Section 3 is devoted to nonlinear estimates. We make extensive use of the ideas of paradifferential calculus of Bony [**1**] to study nonlinear operators acting on the Sobolev spaces defined in chapter 2. We also prove results concerning products or conjugation of an element of such a Sobolev space with an oscillatory exponential. Section 4 gives the proof of the theorem. We perform the method of elimination of oscillatory exponentials outlined above. The main tool is again paradifferential calculus, which allows us to decompose the right hand side of the equation as a sum of a nice term, and of a really oscillating contribution, that we eliminate using the non charactericity of the phase. Since the true equation is (0.1) rather than (0.2), we use weighted versions of the Sobolev spaces defined in chapter 2 to treat the contribution coming from X in the right hand side of (0.1). When all oscillatory contributions have been cancelled, the proof of the theorem, as well as the description of the asymptotics of the solution, follow from standard arguments.