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SPECTRAL PROPERTIES OF SELF-SIMILAR LATTICES AND ITERATION OF RATIONAL MAPS

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Abstract. — In this text we consider discrete Laplace operators defined on lattices based on finitely-ramified self-similar sets, and their continuous analogous defined on the self-similar sets themselves. We are interested in the spectral properties of these operators. The basic example is the lattice based on the Sierpinski gasket. We introduce a new renormalization map which appears to be a rational map defined on a smooth projective variety (more precisely, this variety is isomorphic to a product of three types of Grassmannians: complex Grassmannians, Lagrangian Grassmannian, orthogonal Grassmannians). We relate some characteristics of the dynamics of its iterates with some characteristics of the spectrum of our operator. More specifically, we give an explicit formula for the density of states in terms of the Green current of the map, and we relate the indeterminacy points of the map with the so-called Neumann-Dirichlet eigenvalues which lead to eigenfunctions with compact support on the unbounded lattice. Depending on the asymptotic degree of the map we can prove drastically different spectral properties of the operators. Our formalism is valid for the general class of finitely ramified self-similar sets (*i.e.* for the class of p.c.f. self-similar sets of Kigami). Hence, this work aims at a generalization and a better understanding of the initial work of the physicists Rammal and Toulouse on the Sierpinski gasket.

Résumé **(Propriétés spectrales des réseaux auto-similaires et itération d'applications rationelles)**

Dans ce texte, nous considérons le laplacien discret, défini sur un réseau construit à partir d'un ensemble auto-similaire finiment ramifié, et son analogue continu défini sur l'ensemble auto-similaire lui-même. Nous nous intéressons aux propriétés spectrales de ces opérateurs. L'exemple le plus classique est celui du triangle de Sierpinski (Sierpinski gasket) et du réseau discret associé. Nous introduisons une nouvelle application de renormalisation qui se trouve être une application rationnelle définie sur une variété projective lisse (plus précisément, cette variété est un produit de grassmanniennes de trois types : grassmanniennes classiques, grassmanniennes lagrangiennes, grassmanniennes orthogonales). Nous relions certaines propriétés spectrales de ces opérateurs avec la dynamique des itérés de cette application. En particulier, nous donnons une formule explicite de la densité d'états en termes du courant de Green de l'application, et nous caractérisons le spectre de Neumann-Dirichlet (qui correspond aux fonctions propres à support compact sur l'ensemble infini) à l'aide des points d'indétermination de l'application. Suivant le degré asymptotique de l'application nous pouvons prouver que les propriétés spectrales de l'opérateur sont très différentes. Notre formalisme s'applique à la classe des ensembles auto-similaires finiment ramifiés (ou autrement dit à la classe des « p.c.f. self-similar sets » de Kigami). Ainsi, ce travail généralise et donne une compréhension plus profonde des résulats obtenus initialement par Rammal et Toulouse dans le cas du triangle de Sierpinski.

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INTRODUCTION

In this text we investigate the spectral properties of Laplace operators defined on hierarchical lattices based on finitely ramified self-similar sets, and their continuous analogs. The basic example is the lattice based on the Sierpinski gasket. These operators have much to do with the operators considered in the context of Schrödinger operators with random or quasi-periodic potential. Here, the disorder is not in the potential but in the lattice itself. It is well-known that in the context of Schrödinger operators on the line the spectral properties are intimately related to the dynamics of the propagator of the underlying differential equation (*cf.* for example [**8**], [**33**]). In comparison, in our models we will show that the characteristics of the spectrum of our operator are related to the dynamics of the iterates of a certain renormalization map that we explicitly define and that appears to be a rational self-map of a compact complex manifold.

The interest in such lattices and in their spectral properties comes from physicists (*cf.* [**35**], [**34**], [**1**] and [**4**]) because they present interesting computable models, with peculiar properties. In [**35**], [**34**], on the particular lattice based on the Sierpinski gasket, Rammal and Toulouse discovered interesting relations between the spectrum of the discrete Laplace operator and the dynamics of the iteration of some rational map on $\mathbb{C}$. More precisely, they exhibited a polynomial map on $\mathbb{C}$ that relates the spectrum of the operator on successive scales: they remarked that if λ is an eigenvalue at level $n+1$ then $\lambda(5-\lambda)$ is an eigenvalue at level n. Traditionally, this law was called the spectral decimation of the Sierpinski gasket, *i.e.* this terminology reflects the existence of a 1-dimensional map that relates the spectrum of the operator on successive scales. Starting from this, Rammal ([**34**]) gave a fairly complete description of the spectrum of the discrete operator on this lattice. In particular, he computed explicitly the eigenvalues and showed the existence of the so-called molecular states (that we call Neumann-Dirichlet eigenfunctions in this text) which are eigenfunctions with compact support. This was made rigorous and generalized to the continuous operator defined on the Sierpinski gasket itself by Fukushima and Shima (*cf.* [**19**]). The spectral type of the operator on the Sierpinski lattice, has been analyzed by Teplyaev, *cf.* [**49**].

In general, the spectral decimation that works for the Sierpinski gasket is not valid, and the question of generalizing the initial work of Rammal remained unsolved. In [**20**] a class of lattices for which the spectral decimation works is exhibited. In [**38**], for the particular example of a Sturm-Liouville operator defined on $\mathbb{R}$, the author made explicit some relations between the spectral properties of the operator and the properties of the dynamics of the iterates of a rational map; this map is no longer 1-dimensional but is defined on the 2-dimensional projective space.

This text aims at a generalization of these previous works. Besides the interest of the generalization, this brings new understanding of the models. In particular, the renormalization map involved is now multidimensional and certain notions which are specific to the dynamics in higher dimension and which were hidden in the case of the Sierpinski gasket (where the renormalization map involved was 1-dimensional), such as the notion of indeterminacy points (which corresponds to the singularities of the map), the degree of the iterates, enter the discussion and play an important role. In comparison with our previous work, [**38**], the main progress that allows us to handle the general case is the construction of a new renormalization map. This renormalization map is a rational map defined on some compact Kähler manifold. It is of the type of the maps considered in [**13**], [**12**], and our techniques rely heavily on recent works of Fornaess Sibony, Diller Favre, Guedj (*cf.* [**45**], [**16**], [**13**], [**12**], [**14**]) on the dynamics of rational maps in higher dimensions. It is interesting to note that many of the key notions in this field (such as the degrees of the iterates, the indeterminacy points, the Green current) find a significance related to the spectral properties of our operators. In particular, we are able to give an explicit expression for the density of states in terms of the Green current of the map and we prove that the molecular states of Rammal (called Neumann-Dirichlet eigenvalues in the text) correspond exactly to the indeterminacy points of the map.

Since the text is long, we first describe the model and our results on the particular example of the lattice associated with the Sierpinski gasket. Let $F \subset \mathbb{C}$, $F = \{0, 1, \frac{1}{2} + i\frac{\sqrt{3}}{2}\}$, be the vertices of a unit triangle, and Ψ_1, Ψ_2, Ψ_3 be the three homotheties with ratio $\frac{1}{2}$ and centers the points $0, 1, \frac{1}{2} + i\frac{\sqrt{3}}{2}$, respectively. It is well-known that there exists a unique proper subset X of $\mathbb{C}$ self-similar with respect to Ψ_1, Ψ_2, Ψ_3, *i.e.* such that $X = \cup_{i=1}^{3} \Psi_i(X)$, and that it is the celebrated Sierpinski gasket, represented on Figure 1.

Fix now a sequence $\omega \in \{1, 2, 3\}^{\mathbb{N}}$, called the blow-up, and define $X_{\langle 0 \rangle} = X$ and

$$X_{\langle n \rangle} = \Psi_{w_1}^{-1} \circ \cdots \circ \Psi_{w_n}^{-1}(X).$$

It is clear that $X_{\langle n \rangle}$ is an increasing sequence of sets and that $X_{\langle n+1 \rangle}$ is a scaled copy of X that contains $X_{\langle n \rangle}$ as one of the three subcells; more precisely, we have

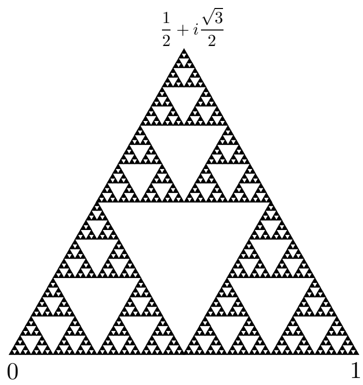

FIGURE 1

$X_{\langle n\rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_{n+1}}^{-1}(\Psi_{\omega_{n+1}}(X))$, which is clearly a subset of $X_{\langle n+1\rangle}$. Remark that the position of the cell $X_{\langle p\rangle}$ in $X_{\langle n\rangle}$ for $n > p$ depends on the blow-up ω. We then set

$$X_{\langle\infty\rangle} = \cup_{n=0}^{\infty} X_{\langle n\rangle}.$$

We define the boundary of $X_{\langle n\rangle}$ by $\partial X_{\langle 0\rangle} = F$ and

$$\partial X_{\langle n\rangle} = \Psi_{w_1}^{-1} \circ \cdots \circ \Psi_{w_n}^{-1}(F).$$

There is a natural discrete sequence of lattices associated with this structure. The lattice at level 0 is $F_{\langle 0\rangle} = F$, the vertices of the unit triangle in $X_{\langle 0\rangle}$. The lattice at level n, is the set of vertices of the unit triangles in $X_{\langle n\rangle}$. More precisely,

$$F_{\langle n\rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(\cup_{j_1,\ldots,j_n} \Psi_{j_1} \circ \cdots \circ \Psi_{j_n}(F)).$$

The position of $F_{\langle 0\rangle}$ in the lattice at level n depends on ω, and we represent on figure 2 the lattice at level 4, F_4. The bolded small triangle is the set $F_{\langle 0\rangle}$ for the blow-up starting from $(\omega_1, \ldots, \omega_4) = (1,1,1,1)$ on the left and $(1,3,1,2)$ on the right. The sequence $F_{\langle n\rangle}$ is increasing and we set

$$F_{\langle\infty\rangle} = \cup_{n=0}^{\infty} F_{\langle n\rangle},$$

and $\partial F_{\langle n\rangle} = \partial X_{\langle n\rangle}$.

It is important to realize that the infinite lattices $F_{\langle\infty\rangle}$ obtained from different blow-ups ω and ω' are a priori not isomorphic (except when ω and ω' are equal after a certain level). To understand this, one can compare the constant blow-up $(1, \ldots, 1, \ldots)$ with a non-stationary blow-up: the first one contains a point with only 2 neighbors (which is the point 0, center of the homothety Ψ_1), on the second one all points have 4 neighbors (indeed, the boundary points $\partial F_{\langle n\rangle}$ are sent to infinity when n goes to infinity).

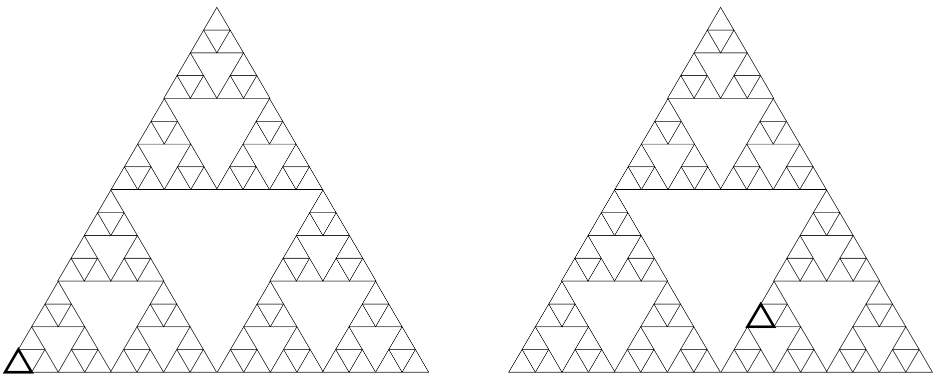

FIGURE 2

The aim of this text is to investigate the spectral properties of some natural Laplace operator defined either on the infinite lattice $F_{\langle\infty\rangle}$ or on the unbounded set $X_{\langle\infty\rangle}$. The class of lattices or self-similar sets we consider is issued from the class of finitely-ramified self-similar sets (also called p.c.f. self-similar sets in [**25**]) described in section 1.1, and is much larger than the Sierpinski gasket. Although the classical examples have a natural geometrical embedding, these sets are defined abstractly from a very simple finite structure: one starts from a finite set F and one constructs $F_{\langle 1\rangle}$ as the union of N copies of F, glued together according to a prescribed rule (represented by an equivalence relation $\mathcal{R}$ on $\{1,\dots,N\}\times F$), then $F_{\langle 2\rangle}$ is defined as the union of N copies of $F_{\langle 1\rangle}$ glued together according to the same rule, and so on. From this discrete structure, one can construct an increasing sequence of sets $F_{\langle n\rangle}$, and also a self-similar set X (*cf.* section 1.2 for precise definitions).

To take into account the eventual symmetries of the picture, we fix a group of symmetries acting on each $F_{\langle n\rangle}$ (but in general not on $F_{\langle\infty\rangle}$). For the Sierpinski gasket we can see that the group $G\sim S_3$ (S_3 denotes the group of permutation of F) of isometries of the regular triangle $\partial F_{\langle n\rangle}$ leaves globally invariant the lattice $F_{\langle n\rangle}$. We fix this group G as the group of symmetries of the structure (*i.e.* this means that we will only consider G-invariant objects).

Note that for consistency with the notations of the main text, we denote by N the number of subcells of $F_{\langle 1\rangle}$. Here, we have $N=3$.

We now define the type of operators we will consider in this text. We restrict to the discrete setting in this introduction and we present the definitions only in the case of the Sierpinski gasket. On $F_{\langle n\rangle}$ we define the difference operator $A_{\langle n\rangle}$ as the operator on $\mathbb{R}^{F_{\langle n\rangle}}$ defined by

$$A_{\langle n\rangle}f(x) = -\sum_{y\sim x}(f(y)-f(x)),\quad \forall f\in\mathbb{R}^{F_{\langle n\rangle}}, \tag{1}$$