MÉMOIRES DE LA SMF 94

ON MAPPING PROPERTIES OF THE GENERAL RELATIVISTIC CONSTRAINTS OPERATOR IN WEIGHTED FUNCTION SPACES, WITH APPLICATIONS

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Société Mathématique de France 2003
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification*. — 83C05.

***Key words and phrases*. —** General relativistic initial data, non-connected black holes, asymptotically simple space-times, initial data gluing.

P.C.: Partially supported by a Polish Research Committee grant 2 P03B 073 24.
E.D.: Partially supported by the ACI program of the French Ministry of Research.

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Abstract. — Generalizing an analysis of Corvino and Schoen, we study surjectivity properties of the constraint map in general relativity in a large class of weighted function spaces. As a corollary we prove several perturbation, gluing, and extension results: we show existence of non-trivial, singularity-free, vacuum space-times which are *stationary* in a neighborhood of i^0; for small perturbations of *parity-covariant* initial data sufficiently close to those for Minkowski space-time this leads to space-times with a smooth global $\mathscr{I}$; we prove existence of initial data for many black holes which are exactly Kerr — or exactly Schwarzschild — both near infinity and near each of the connected components of the apparent horizon; under appropriate conditions we obtain existence of vacuum extensions of vacuum initial data across compact boundaries; we show that for generic metrics the deformations in the Isenberg-Mazzeo-Pollack gluings can be localized, so that the initial data on the connected sum manifold coincide with the original ones except for a small neighborhood of the gluing region; we prove existence of asymptotically flat solutions which are static or stationary up to r^{-m} terms, for any fixed m, and with multipole moments freely prescribable within certain ranges.

Résumé **(Sur les propriétés de l'opérateur de contraintes relativistes dans des espaces à poids, et applications)**

Nous étudions les propriétés de surjectivité de l'application de contraintes en relativité générale dans une large classe d'espaces fonctionnels à poids, généralisant ainsi une analyse de Corvino et Schoen. Comme corollaire on obtient plusieurs résultats de perturbation, de recollement, ou d'extension. Ainsi, nous démontrons l'existence d'espaces-temps non triviaux, sans singularités, solutions d'équations d'Einstein du vide, qui sont *stationnaires* dans un voisinage de i^0. Pour des données initiales proches de celles de Minkowski ceci conduit, sous une condition de parité approximative, à des espaces-temps avec un infini isotrope $\mathscr{I}$ global et lisse. Nous prouvons l'existence de données initiales pour des trous noirs multiples qui sont exactement kerriennes, ou exactement schwarzschildiennes, dans une région asymptotique, mais aussi près de chaque composante connexe de l'horizon apparent. Nous montrons que pour des métriques génériques les perturbations des données initiales introduites par les recollements du type Isenberg-Mazzeo-Pollack peuvent être localisées, de sorte que les données initiales sur la variété obtenue en prenant la somme connexe coincident avec les données initiales originelles, sauf dans un petit voisinage de la zone de recollement. Nous prouvons l'existence de solutions asymptotiquement plates qui sont statiques ou stationnaires modulo des termes en r^{-m}, avec m arbitrairement prescrit, et avec des moments multipolaires qu'on peut spécifier librement dans certains ouverts.

CONTENTS

CHAPTER 1

INTRODUCTION

In a recent significant paper [**30**] Corvino has presented a new gluing construction of scalar flat metrics, leading to the striking result of existence of non-trivial scalar flat metrics which are exactly Schwarzschildian at large distances; see also [**33**]. Extensions of the results in [**30**] have been announced in [**31**], and those results should be available[(1)] in a near future [**32**]. A reading of the proofs in [**30**] reveals that the arguments there can be simplified or streamlined using known techniques for PDE's in weighted Sobolev spaces (*cf., e.g.* [**1, 3, 7, 16, 41, 52**]). Further, the methods introduced by Corvino and Schoen can be applied in other contexts to obtain new classes of solutions of the general relativistic constraint equations. The object of this paper is to present an abstract version, in a large class of weighted Sobolev spaces, of the arguments of Corvino and Schoen. Specific results on compact manifolds with boundary (as considered by Corvino), or on asymptotically flat manifolds, or on asymptotically hyperboloidal manifolds, can then be obtained by an appropriate choice of the weight functions. More precisely, we develop a general theory of mapping properties of the solutions of the linearized constraint operator in a class of weighted Sobolev spaces, assuming certain inequalities. The class of weighted Sobolev spaces includes those of Christodoulou — Choquet-Bruhat [**16**], appropriate in the asymptotically Euclidean context, as well as an exponentially weighted version thereof, and distance–weighted spaces near a boundary, or an exponentially weighted version thereof; the latter two classes are relevant near a compact boundary, or in an asymptotically hyperboloidal context. We establish the required inequalities in all the spaces just mentioned. An appropriate version of the inverse function theorem allows one to produce new classes of solutions of interest. One application is that of existence of space-times which are Kerrian near spatial infinity; this has already been observed in [**31**]. We apply our techniques to produce two further large classes of initial data sets with controlled asymptotic behavior at spatial infinity. The first class is obtained by gluing any

(1)This paper has been written after [**30, 31**], but independently of [**32**].

asymptotically flat initial data with data in the exterior region which are exactly stationary there. This leads to a large class of space-times which are exactly stationary away from the domain of influence of a compact set. The second class consists of initial data which are approximately stationary in the asymptotic region, with the non-stationary part decaying at a prescribed (as high as desired) order in terms of powers of r. On the other hand the stationary part is controlled by a set of multipole moments which are freely prescribable within certain ranges. Such initial data are relevant to the program of [**37, 39**]. Yet another application is an extension result for initial data near the Minkowskian ones, which leads to asymptotically simple space-times, or to new "many black hole" space-times. Our final application here is a gluing construction for generic CMC initial data sets, in which the perturbation of the metric is localized in a small neighborhood of the points where the gluing is performed. This makes use of, and refines, the recent gluing construction of Isenberg, Mazzeo and Pollack [**45, 46**]. Some further applications, involving local extensions near positively or negatively curved space forms, or concerning the construction of initial data with controlled Bondi functions, will be discussed elsewhere.

We note that all the results in Section 3 are valid when M is a compact manifold without boundary by setting all the weight functions to one, $\varphi = \phi = \psi = 1$, and by taking the compact set $\mathscr{K}$ appearing in Proposition 3.1 and elsewhere equal to M.

Acknowledgements. — We thank R. Beig, J. Corvino, H. Friedrich and W. Simon for useful comments or discussions, as well as a referee for detailed criticism.

CHAPTER 2

THE CONSTRAINTS MAP

The aim of this section is to establish some algebraic-differential properties of the constraints map, and some elementary properties of the associated differential operators in a class of weighted Sobolev spaces. The reader is referred to Appendix A for the definition of the latter.

Initial data (g, K) for the vacuum Einstein equations belong to the zero level set of the *constraints map*:

$$\begin{pmatrix} J \\ \\ \rho \end{pmatrix} (K,g) := \begin{pmatrix} 2(-\nabla^j K_{ij} + \nabla_i \operatorname{tr} K) \\ \\ R(g) - |K|^2 + (\operatorname{tr} K)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}. \tag{2.1}$$

These are the general relativistic constraint equations whatever the space-dimension n. As Equations (2.1) are trivial in space-dimension zero and one, in the remainder of this paper we shall assume that $n \geqslant 2$.

Let $h = \delta g$ and $Q = \delta K$, the linearization of the constraints map at (K, g) reads

$$P(Q,h) = \begin{pmatrix} -K^{pq}\nabla_i h_{pq} + K^q{}_i(2\nabla^j h_{qj} - \nabla_q h^l{}_l) \\ \qquad -2\nabla^j Q_{ij} + 2\nabla_i \operatorname{tr} Q - 2(\nabla_i K^{pq} - \nabla^q K^p{}_i)h_{pq} \\ \\ -\Delta(\operatorname{tr} h) + \operatorname{div}\operatorname{div} h - \langle h, \operatorname{Ric}(g)\rangle + 2K^{pl}K^q{}_l h_{pq} \\ \qquad -2\langle K, Q\rangle + 2\operatorname{tr} K(-\langle h, K\rangle + \operatorname{tr} Q) \end{pmatrix}. \tag{2.2}$$

Remark 2.1. — We note that for any real numbers a and b it holds

$$P(aK, bg) = \begin{pmatrix} (a-b)J(K,g) \\ \\ -bR(g) + 2(b-a)[|K|^2 - (\operatorname{tr} K)^2] \end{pmatrix}. \tag{2.3}$$

The order of the differential operators that appear in P is

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

which can be written in the Agmon-Douglis-Nirenberg form (*cf., e.g.* [**54**, p. 210])

$$\begin{pmatrix} s_1+t_1 & s_1+t_2 \\ s_2+t_1 & s_2+t_2 \end{pmatrix},$$

with $s_1=-1$, $s_2=0$, $t_1=t_2=2$; here it is understood that an operator of order 0 is also an operator of order 2 with vanishing coefficients in front of the first and second derivatives. It follows that the symbol P' of the principal part of P in the sense of Agmon-Douglis-Nirenberg reads

$$P'(x,\xi)(Q,h) = \begin{pmatrix} 2(-\xi^s\delta^t_i + \xi_i g^{st}) & -K^{pq}\xi_i + 2K^q{}_i\xi^p - K^l{}_i\xi_l g^{pq} \\ 0 & -|\xi|^2 g^{pq} + \xi^p\xi^q \end{pmatrix} \begin{pmatrix} Q_{st} \\ h_{pq} \end{pmatrix},$$

while the formal L^2-adjoint of P takes the form

(2.4)
$$P^*(Y,N) = \begin{pmatrix} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l g_{ij} - K_{ij}N + \operatorname{tr} K\ N g_{ij}) \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l\nabla_q Y^l g_{ij} - \Delta N g_{ij} + \nabla_i\nabla_j N \\ \quad +(\nabla^p K_{lp} g_{ij} - \nabla_l K_{ij})Y^l - N\mathrm{Ric}\,(g)_{ij} + 2NK^l{}_i K_{jl} - 2N(\operatorname{tr}\ K)K_{ij} \end{pmatrix}.$$

From this we obtain the Agmon-Douglis-Nirenberg symbol $P^{*\prime}$ of the principal part of P^*,

$$P^{*\prime}(x,\xi)(Y,N) = \begin{pmatrix} 2(\xi_{(i}\delta^l_{j)} - \xi^l g_{ij}) & 0 \\ K_{ij}\xi^l - 2K^l{}_{(i}\xi_{j)} + K^{pl}\xi_q g_{ij} & \xi_i\xi_j - |\xi|^2 g_{ij} \end{pmatrix} \begin{pmatrix} Y_l \\ N \end{pmatrix}. \tag{2.5}$$

Remark 2.2. — Recall that the formal adjoint P^* is defined by the requirement that for all smooth (Q,h)'s and for all compactly supported smooth (Y,N)'s we have

$$\langle P^*(Y,N),(Q,h)\rangle_{L^2(g)\oplus L^2(g)} = \langle (Y,N), P(Q,h)\rangle_{L^2(g)\oplus L^2(g)}.$$

It is easily seen by continuity and density arguments that this equation still holds for[(1)] all $(Q,h)\in H^1_{\mathrm{loc}}\times H^2_{\mathrm{loc}}$ and for all $(Y,N)\in \mathring{H}^1_{\phi,\psi}\times\mathring{H}^2_{\phi,\psi}$.

We wish to check ellipticity of PP^*, for this we need the following:

Lemma 2.3. — *Suppose that* $\dim M \geqslant 2$, *then* $P^{*\prime}(x,\xi)$ *is injective for* $\xi\neq 0$.

Proof. — We define a linear map α from the space S_2 of two-covariant symmetric tensors into itself by the formula

$$\alpha(S) = S - (\operatorname{tr} S)g. \tag{2.6}$$

Let $\xi\neq 0$, if (Y,N) is in the kernel of $P^{*\prime}(x,\xi)$ then

$$\alpha(\xi_{(i}Y_{j)}) = 0,$$

so that $\xi_{(i}Y_{j)}=0$, and $Y=0$. It follows that

$$\alpha(\xi_i\xi_j)N = 0,$$

which implies $N=0$. □

(1) See Appendix A for the definitions of the function spaces we use.