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NICHOLAS HANGES
FRANÇOIS TREVES

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ON THE LOCAL HOLOMORPHIC EXTENSION OF CR FUNCTIONS

NICHOLAS HANGES† AND FRANÇOIS TREVES‡

0. Introduction. The history of the holomorphic continuation, across a hypersurface Σ, of functions defined and holomorphic on one side of Σ goes back to the discovery of strong pseudoconvexity — and to the proof in Levi [1] that, in a strongly pseudoconvex domain of $\mathbb{C}^2$ with smooth boundary Σ, there are holomorphic functions that have no holomorphic extension across the boundary, to the concave side. Later, in Lewy [1], it was shown that every holomorphic function on the concave side extends to the convex side. It is now customary to rephrase such results in the language of CR functions on Σ, and of their germs at a point: *on a strongly pseudoconvex hypersurface in* $\mathbb{C}^N$ *there are germs of CR functions that do not extend to the pseudoconcave side* (Levi); *every germ of CR function extends to the pseudoconvex side* (Lewy).

The question of the extension (always, for us, holomorphic extension) of germs is radically different from that of the extension of CR functions defined in the whole boundary Σ. Extension of the latter kind can be viewed as an aspect of the *Hartogs phenomenon*. Let us recall how. Let $\Omega \subset \mathbb{C}^N (N \geq 2)$ be an open and bounded set, whose complement consists of a single, unbounded, connected component, and whose boundary Σ is fairly smooth, say of class $\mathcal{C}^2$. Let h be a function defined and $\mathcal{C}^2$ in the whole of Σ. Provided h satisfies the tangential Cauchy-Riemann equations, one can find a $\mathcal{C}^2$ extension $\tilde{h}$ to Ω such that $\bar\partial\tilde{h}$, as well as its first partial derivatives, vanish on Σ. Let $g = \bar\partial\tilde{h}$ in Ω, $g \equiv 0$ in $\mathbb{C}^N\backslash\Omega$; we have $g \in \mathcal{C}^1(\mathbb{C}^N)$ and $\bar\partial g = 0$. We can then solve $\bar\partial u = g$ in $\mathbb{C}^N$, with $u \in \mathcal{C}^1(\mathbb{C}^N)$ and $u \equiv 0$ in $\mathbb{C}^N\backslash\Omega$ (since $N \geq 2$). Clearly, $\tilde{h} - u$ is holomorphic in Ω and is equal to h on Σ. The extension of the globally defined CR function h depends on the *topology* of Σ; its *geometry*, *e.g.*, whether Σ has convex or concave parts, is irrelevant.

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The situation is quite different when one tries to extend the germs of CR functions.

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1. The Hypersurface Case.

First of all, we recall the definitions and fix the notation. We consider a real hypersurface Σ in $\mathbb{C}^{n+1}(n \geq 1)$, of class $\mathcal{C}^2$ (this will always be our smoothness hypothesis, unless specified otherwise). Call $z_1, ..., z_n$ and w the complex coordinates in $\mathbb{C}^{n+1}$. We shall assume that $0 \in \Sigma$ and that the tangent hyperplane to Σ at 0 is the hyperplane $\Im w = 0$. In other words, we are going to assume that, in an open ball $\Omega \subset \mathbb{C}^{n+1}$ centered at the origin, Σ is defined by an equation

$$\Im w = \phi(z, \Re e w), \tag{1.1}$$

with ϕ real-valued and of class $\mathcal{C}^2$ in the closure of Ω, and

$$\phi|_0 = 0, d\phi|_0 = 0. \tag{1.2}$$

The pullbacks to Σ of the differentials $dz_i (1 \leq i \leq n)$, dw span a vector subbundle, here denoted by $T'^{1,0}\Sigma$, of the complexified tangent bundle $\mathbb{C}T^*\Sigma$; the rank of $T'^{1,0}\Sigma$ is equal to $n+1$. Its orthogonal in $\mathbb{C}T\Sigma$ is the vector subbundle of rank n, $T^{0,1}\Sigma$, spanned by the vector fields tangent to Σ that are linear combinations of $\partial/\partial \bar{z}_i (1 \leq i \leq n)$ and $\partial/\partial \bar{w}$. By a CR function h on Σ we shall mean a continuous function in Σ such that $Lh = 0$ whatever the $\mathcal{C}^1$ section L of $T^{0,1}\Sigma$. The equation $Lh = 0$ can, and must, be understood in the distribution sense.

As our standpoint is strictly local, we may as well assume that $\Sigma = \Sigma \cap \Omega$, and that the hypersurface Σ subdivides Ω into two *sides*: Ω^+, in which $\Im w > \phi(z, \Re e w)$; Ω^-, where $\Im w < \phi(z, \Re e w)$. The boundary value of any continuous function in $\Omega^+ \cup \Sigma$ (i.e., "continuous up to the boundary") that is holomorphic in Ω^+ is a CR function on Σ. The problem we wish to discuss is the localized converse: to seek properties of Σ which ensure that every (continuous) CR function h in a neighborhood of 0 in Σ is the boundary value $bv\tilde{h}$, say in the distribution sense, and possibly in a smaller neighborhood, of a holomorphic function $\tilde{h}$ in a set $\tilde{V} \cap \Omega^+$ with $\tilde{V}$ some open neighborhood of 0 in $\mathbb{C}^{n+1}$. If the latter is true, we say that the germ of h *extends to* Ω^+. We recall that the fact that $\tilde{h}$ has a distribution boundary value, is equivalent to the property that, in $\tilde{V}, \tilde{h}$ grows slowly at the *edge* $\tilde{V} \cap \Sigma$, i.e., to each compact subset K of $\tilde{V}$ there is an integer $k > 0$ and a constant $C > 0$ such that

$$|\tilde{h}(z)| \leq C \operatorname{dist}(z, \Sigma)^{-k}, \ \forall z \in K \cap \Omega^+.$$

(Here the variable is denoted by z rather than (z, w).)

We now recall a number of basic facts, all fairly elementary, and most well known; and first of all, the property of "unique continuation" across a hypersurface:

Proposition 1.1. *If the germ of a CR function at* 0, *on* Σ, *extends to both sides* Ω^+ *and* Ω^-, *then it is the restriction to* Σ *of the germ of a holomorphic function in* $\mathbb{C}^{n+1}$.

The next observation is a particular case of a more general statement (see e.g., Baouendi-Jacobowitz-Treves [1], Lemma 2.4).

Proposition 1.2. *Suppose a holomorphic function* $\tilde{h}$ *in* $\tilde{V} \cap \Omega^+$, *with* $\tilde{V}$ *an open neighborhood of* 0 *in* $\mathbb{C}^{n+1}$, *has a boundary value* h *on* $V = \tilde{V} \cap \Sigma$ *in the distribution sense. If* h *is a continuous function in* V, *then* $\tilde{h}$ *is continuous in* $V \cup \Omega^+$.

The next statement is a direct consequence of the Baire category theorem (cf. Lemma III.5.1 in Treves [1]):

Proposition 1.3. *Let* U *be an open neighborhood of* 0 *in* Σ *with the property that to each CR function* $h \in C^0(U)$ *there is an open neighborhood* $\tilde{V}$ *of* 0 *in* $\mathbb{C}^{n+1}$ *such that* h *extends holomorphically to* $\tilde{V} \cap \Omega^+$. *Then* $\tilde{V}$ *can be chosen independently of* h.

Extension of (germs at 0 of) CR functions to one side, say Ω^+, is the same as extension to a full neighborhood of 0 in $\mathbb{C}^{n+1}$ (to the germ $(\mathbb{C}^{n+1}, 0)$) of (germs at 0 of) holomorphic functions in Ω^-. This is a consequence of the classical decomposition of CR functions (see Andreotti and Hill [1]):

Proposition 1.4. *If* h *is a continuous CR function in an open neighborhood* U *of* 0 *in* Σ, *then there are an open neighborhood* $\tilde{V} \subset \Omega$ *of* 0 *in* $\mathbb{C}^{n+1}$, *and holomorphic functions* $\tilde{h}^+$ *and* $\tilde{h}^-$ *in* $\tilde{V} \cap \Omega^+$ *and* $\tilde{V} \cap \Omega^-$ *respectively, such that* $h = bv\, \tilde{h}^+ - bv\, \tilde{h}^-$ *in* $\tilde{V} \cap \Sigma$.

Below we shall also make use of the following approximation results:

Proposition 1.5. *To each open neighborhood $\tilde{V} \subset \Omega$ of 0 in $\mathbb{C}^{n+1}$ there is another open neighborhood $\tilde{V}_0 \subset \tilde{V}$ such that every continuous function in $\tilde{V} \cap (\Omega^+ \cup \Sigma)$ which is holomorphic in $\tilde{V} \cap \Omega^+$ is the uniform limit, in $\tilde{V}_0 \cap (\Omega^+ \cup \Sigma)$, of a sequence of holomorphic polynomials.*

Remark: By a simple translation argument the following variant of Proposition 1.5 is also true. To each open neighborhood $\widetilde{V} \subset \Omega$ of 0 in $\mathbb{C}^{n+1}$ there is another open neighborhood $\widetilde{V}_0 \subset \widetilde{V}$ such that every function which is holomorphic in $\widetilde{V} \cap \Omega^+$ is the uniform limit, on compact subsets of $\widetilde{V}_0 \cap \Omega^+$, of a sequence of holomorphic polynomials.

Proposition 1.6. *To each open neighborhood U of O in Σ there is another open neighborhood $U_0 \subset U$ of O such that every continuous CR function in U is the uniform limit, in U_0, of a sequence of holomorphic polynomials.*

Prop.1.5 is stated, and proved, as Th. V.7.2 in Treves [1]. Prop.1.6 is a direct consequence of the approximation formula in Baouendi-Treves [1] (also Th. II.2.1 in Treves [1]). [The authors would like to thank J. P. Rosay for pointing out an embarassing mistake in an earlier version of the article, specifically in an attempt to derive directly Prop.1.6 from Prop.1.5.]

Propositions 1.3 and 1.6 have the following consequence:

Proposition 1.7. *The following properties are equivalent:*

(1.3) *To each open neighborhood U of 0 in Σ there is an open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that every CR function $h \in C^0(U)$ extends holomorphically to $\tilde{V} \cap \Omega^+$.*

(1.4) *To each open neighborhood U of 0 in Σ there is an open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that any holomorphic polynomial that vanishes in $\tilde{V} \cap \Omega^+$ also vanishes in U.*

Proof. Let U and $\tilde{V}$ be as in (1.3). If the polynomial P does not vanish in U, $1/P$ extends holomorphically to $\tilde{V} \cap \Omega^+$, and therefore P cannot vanish there.

Let now U and U_0 be related as in Prop.1.6. There is an open (and bounded) neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that, given any holomorphic polynomial P and any $(z,w) \in \tilde{V} \cap \Omega^+$, then $P(z_0, w_0) = P(z,w)$ for some $(z_0, w_0) \in U_0$. This entails that if a sequence of holomorphic polynomials P_ν converges in $C^0(U_0)$ to a CR function h then P_ν converges uniformly in $\tilde{V} \cap \Omega^+$, to a holomorphic function h whose boundary value on U_0 is equal to h. $\square$

Prop. 1.5 has also the following consequence, which will be of use later.