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FIXED POINT THEORY AND TRACE FOR BICATEGORIES

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Abstract. — The Lefschetz fixed point theorem follows easily from the identification of the Lefschetz number with the fixed point index. This identification is a consequence of the functoriality of the trace in symmetric monoidal categories.

There are refinements of the Lefschetz number and the fixed point index that give a converse to the Lefschetz fixed point theorem. An important part of this theorem is the identification of these different invariants.

We define a generalization of the trace in symmetric monoidal categories to a trace in bicategories with shadows. We show the invariants used in the converse of the Lefschetz fixed point theorem are examples of this trace and that the functoriality of the trace provides some of the necessary identifications. The methods used here do not use simplicial techniques and so generalize readily to other contexts.

***Résumé* (Théorie du point fixe et trace pour les bicatégories).** — Le théorème du point fixe de Lefschetz découle facilement de l'identification du nombre de Lefschetz avec l'indice de point fixe. Cette identification est une conséquence de la fonctorialité de la trace dans les catégories symétriques monoïdales.

Ce sont des raffinements du nombre de Lefschetz et de l'indice de point fixe qui fournissent la réciproque du théorème du point fixe de Lefschetz. Une partie importante de ce théorème est l'identification de ces invariants.

Nous définissons une généralisation de la trace dans les catégories symétriques monoïdales, en une trace dans les bicatégories avec ombres. Nous montrons que les invariants utilisés dans la réciproque du théorème du point fixe de Lefschetz sont des exemples de cette trace, et que la fonctorialité de la trace fournit certaines identifications nécessaires. Les méthodes présentées ici n'utilisent pas de technique simpliciale et peuvent donc être généralisées facilement dans d'autres contextes.

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INTRODUCTION

There are many approaches to determining when a continuous endomorphism of a topological space has a fixed point. One of the simplest is given by the Lefschetz fixed point theorem.

***Theorem A* (Lefschetz fixed point theorem).** — *Let M be a compact ENR and $f \colon M \to M$ be a continuous map. If f has no fixed points then the Lefschetz number of f is zero.*

The Lefschetz number of a map is defined using rational homology and so is relatively easy to compute. Further, if M is a simply connected closed smooth manifold of dimension at least three then a converse to the Lefschetz fixed point theorem also holds.

Theorem B. — *Let $f \colon M \to M$ be a continuous map of a simply connected closed smooth manifold of dimension at least three. Then the Lefschetz number of f is zero if and only if f is homotopic to a map with no fixed points.*

Note that we have replaced 'the map f has no fixed points' with 'the map f is homotopic to a map with no fixed points'. This change only reflects the fact that the Lefschetz number is defined using homology and so cannot distinguish between homotopic maps. In particular, the Lefschetz number cannot determine if a map has no fixed points, it can only determine if it is homotopic to a map with no fixed points.

Unfortunately, Theorem B does not hold if we remove the hypothesis that the space is simply connected. However, by sacrificing some of the computability we can refine the Lefschetz number to an invariant, called the Nielsen number, that detects if the map has fixed points.

Theorem C. — *Let $f \colon M \to M$ be a continuous map of a closed smooth manifold of dimension at least three. The Nielsen number of f, $N(f)$, is the minimum number of fixed points among all maps homotopic to f. In particular, $N(f)$ is zero if and only if f is homotopic to a map with no fixed points.*

The idea behind the Nielsen number is to incorporate information about the fundamental group into the invariant itself. This additional information corresponds to recording which fixed points can be eliminated by a homotopy of the original map.

The Nielsen number is not the most convenient description of this information for defining generalizations of this invariant to other categories and for proving results about relationships between the Nielsen number and basic topological constructions such as cofiber sequences or products. The invariant that retains the necessary information is called the Reidemeister trace. This invariant was defined by Wecken and Reidemeister in **[40, 45]**. It can be used to prove a theorem similar to Theorem C.

Theorem D. — *Let $f\colon M \to M$ be a continuous map of a closed smooth manifold of dimension at least three. The Reidemeister trace of f is zero if and only if f is homotopic to a map with no fixed points.*

Classically, all four of these results were proved using simplicial techniques. In **[11]**, Dold and Puppe proposed an alternative approach. Their idea was to focus on the identification of the Lefschetz number, which is a global invariant, with a local invariant, the fixed point index. It is immediate from the definition that the fixed point index is zero for a map that has no fixed points or is homotopic to a map with no fixed points. Using this observation, the Lefschetz fixed point theorem is a consequence of the identification of the Lefschetz number with the index.

Dold and Puppe approached this identification by defining a more general construction that includes both of these invariants as special cases. Their construction is a 'trace' in any symmetric monoidal category. In some cases the trace is functorial. Dold and Puppe showed that the identification of the Lefschetz number with the index is an example of this functoriality.

In addition to giving an alternate proof of the Lefschetz fixed point theorem, Dold and Puppe's definition of trace can be used to describe generalizations of the fixed point index to other categories. If $f\colon X \to X$ and $p\colon X \to B$ are continuous maps such that $p \circ f = p$ we say that f is a fiberwise map. In **[8]**, Dold defined an index for fiberwise maps and showed that the index is zero for a map that is fiberwise homotopic to a map with no fixed points. The fiberwise index is an example of the trace in symmetric monoidal categories.

It is possible to prove results for the trace in symmetric monoidal categories that can be applied to the special cases of the Lefschetz number and the index. For example, the Lefschetz number and the index are both additive on cofiber sequences. This follows from the additivity of the trace in (some) symmetric monoidal categories, see **[32]**.

Unfortunately, the trace in symmetric monoidal categories cannot be used to describe the invariants of Theorems C and D. Invariants that include information about the fundamental group do not fit into a symmetric monoidal category. However, by replacing symmetric monoidal categories by appropriate bicategories and similarly modifying the definition of the trace we can accommodate these invariants.

Here we implement this philosophy. First we show that the Reidemeister trace is an example of a more general trace. This trace is defined here and is a trace in bicategories with some additional structure; these bicategories are called bicategories with shadows. Just as the Lefschetz number can be identified with the fixed point index, there is more than one description of the Reidemeister trace. There are generalizations of the fixed point index, defined by Reidemeister and Wecken, and of the Lefschetz number, defined by Husseini in **[19]**. Both of these invariants are examples of the trace in bicategories with shadows, and the functoriality of the trace can be used to identify them. There is also an invariant defined by Klein and Williams in **[25]** that can be identified with another example of the trace in a bicategory with shadows.

Next we show that this change in perspective gives definitions and proofs that generalize more easily than the classical approaches. One element of the classical invariants that causes problems for equivariant and fiberwise generalizations is the role played by a base point. Both classical definitions of the Reidemeister trace require that a base point be chosen, but a different choice of the base point does not change the invariant. Modified forms of the Reidemeister trace can be defined without a base point. We show that these invariants are also examples of trace in bicategories, and we use the formal structure of the trace to show that these unbased invariants can be identified with the classical invariants.

The second source of problems for generalizations is only obvious when trying to prove a converse to the Lefschetz fixed point theorem like Theorem D. In **[41]**, Scofield defined a generalization of the Nielsen number to fiberwise maps and gave an example that showed this invariant does not give a converse to the fiberwise Lefschetz fixed point theorem. More recently, Klein and Williams have defined a fiberwise invariant that does give a converse to the fiberwise Lefschetz fixed point theorem.

Theorem E. — *Let $M \to B$ be a fiber bundle with closed smooth manifold fibers F such that $\dim(F) - 3 \geq \dim(B)$. A fiberwise map $f\colon M \to M$ is fiberwise homotopic to a map with no fixed points if and only if the fiberwise Reidemeister trace of f is zero.*

There is another invariant, defined by Crabb and James in **[6]**, that can help to explain the discrepancy between Scofield's invariant and Klein and Williams' invariant. The invariant defined by Crabb and James is a derived form of the Reidemeister trace and so in the transition from a classical invariant to a fiberwise invariant it is sensitive to information that the other forms of the Reidemeister trace, like Scofield's invariant, miss. Crabb and James' invariant can be identified with the invariant defined by Klein and Williams. Crabb and James' invariant, in both its classical and fiberwise forms, is an example of the trace in bicategories with shadows.

More concretely, our goal is to convert Dold and Puppe's outline for proving Theorem A into an approach for proving Theorems D and E. Dold and Puppe's proof identified the Lefschetz number and the fixed point index and then used the observation that the index is zero for maps with no fixed points. Our first step is the same. We start by identifying the form of the Reidemeister trace defined by Husseini with Reidemeister and Wecken's form of the Reidemeister trace. Unfortunately, it is not obvious that Reidemeister and Wecken's form of the Reidemeister trace is zero only when the map is homotopic to a map with no fixed points. The next step in our proof is to identify Reidemeister and Wecken's form of the Reidemeister trace with Crabb