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# MEROMORPHIC QUOTIENTS FOR SOME HOLOMORPHIC G-ACTIONS

BY DANIEL BARLET

ABSTRACT. — Using mainly tools from previous articles we give necessary and sufficient conditions on the $G$-orbits' configuration in $X$ in order that a holomorphic action of a connected complex Lie group $G$ on a reduced complex space $X$ admits a *strongly quasi-proper meromorphic quotient*. To show how these conditions can be used, we show, when $G = K.B$ with $B$ a closed connected complex subgroup of $G$ and $K$ a real compact subgroup of $G$, the existence of a strongly quasi-proper meromorphic quotient for the $G$-action on $X$, assuming a slightly stronger condition than the existence of such a quotient for the $B$-action. We also give a similar result when the connected complex Lie group has the form $G = K.A.K$ where $A$ is a closed connected complex subgroup and $K$ is a compact (real) subgroup.

RÉSUMÉ (*Quotients méromorphes pour certaines $G$-actions holomorphes*). — En utilisant les résultats de précédents articles, nous donnons des conditions nécessaires et suffisantes sur la configuration des $G$-orbites dans $X$ pour que l'action holomorphe d'un groupe de Lie complexe connexe sur un espace complexe réduit $X$ admette un *quotient méromorphe fortement quasi-propre*. Pour illustrer l'intérêt de ces conditions, nous montrons, quand $G = K.B$ où $B$ est un sous-groupe connexe complexe fermé et

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$K$ un sous-groupe compact réel de $G$, l'existence d'un quotient méromorphe fortement quasi-propre pour l'action de $G$ sur $X$ sous une hypothèse légèrement plus forte que l'existence d'un tel quotient pour l'action de $B$ sur $X$. Nous donnons également un résultat analogue quand $G = K.A.K$ où $A$ est un sous-groupe complexe fermé et connexe et $K$ un sous-groupe compact réel de $G$.

## 1. Introduction

In this article we explain how the tools developed in [9], [1], [2] and [3] can be applied to produce, in suitable cases, a meromorphic quotient of a holomorphic action of a connected complex Lie group $G$ on a reduced complex space $X$. This uses the notion of *strongly quasi-proper map* introduced in *loc. cit.* and our first goal is to give three hypotheses, called [H.1], [H.2], [H.3], on the $G$-orbits' configuration in $X$ which are *equivalent* to the existence of a *strongly quasi-proper meromorphic quotient*, notion defined in the Section 1.2.

The proof of this equivalence is the content of Proposition 2.7.1 and Theorem 2.8.1. Then we give a sufficient condition [H.1str], asking the existence of a $G$-invariant set $\Omega_1 \subset X$ which is dense, Zariski open and "good" for the action, to satisfy the condition [H.1].

Note that the conditions [H.1], [H.2], [H.3] introduced in Section 2.7 only depend on the $G$-orbits' configuration in $X$, but the condition [H.1str] depends on the action of $G$ on $X$ itself.

The existence theorem for a strongly quasi-proper meromorphic quotient under our three assumptions is applied to prove the following result:

THEOREM 1.0.1. — *Assume that we have a holomorphic action of a connected complex Lie group $G$ on a reduced complex space $X$. Assume that $G = K.B$ where $K$ is a compact (real) subgroup of $G$ and $B$ a connected complex closed subgroup of $G$. Assume that the action of $B$ on $X$ satisfies the condition* [H.1str] *on a $G$-invariant Zariski open dense subset $\Omega$ in $X^{(1)}$, and the conditions* [H.2] *and* [H.3]. *Then the $G$-action satisfies* [H.1str], [H.2] *and* [H.3]*; so it has a strongly quasi-proper meromorphic quotient.*

A first variant of this result is given by the following theorem.

THEOREM 1.0.2. — *Assume that we have a holomorphic action of a connected complex Lie group $G$ on a reduced complex space $X$. Assume that $G = K.B$ where $K$ is a compact (real) subgroup of $G$ and $B$ a connected complex closed subgroup of $G$. Assume that $K$ normalizes $B$ and that the $B$-action satisfies*

1. This precisely means that there exists a $G$-invariant dense Zariski open set in $X$ which is a "good open set" for the $B$-action (see Section 2.5)

*the conditions* [H.1str]*,* [H.2] *and* [H.3]*. Then the $G$-action satisfies the conditions* [H.1str], [H.2] *and* [H.3] *and so has a strongly quasi-proper meromorphic quotient.*

Here is a second result obtained by a similar method.

THEOREM 1.0.3. — *Let $G$ be a complex connected Lie group and assume that there exists a closed connected complex subgroup $A$ and a compact (real) subgroup $K$ such That $G = K.A.K$. Assume the we have a completely holomorphic action of $G$ on an irreducible complex space $X$ and that the action of $A$ on $X$ satisfies the following properties:*

i) *The hypothesis* [H.1str] *for the $A$-action is satisfied on a $G$-invariant (Zariski good) open set $\Omega_1$ in $X$.*
ii) *The hypothesis* [H.2] *for the $A$-action is satisfied on a $G$-invariant open set $\Omega_0 \subset \Omega_1$ in $X$.*
iii) *The hypothesis* [H.3] *holds for the $A$-action.*

*Then* [H.1str]*,* [H.2] *and* [H.3] *hold for the action of $G$ on $X$. So there exists a SQP meromorphic quotient of $X$ for the $G$-action.*

Of course the hypothesis $G = K.A.K$ is more "general" than the case $G = K.B$. But the hypothesis of this last theorem is more restrictive for the action on $X$ of the closed connected complex subgroup $A$ of $G$: we ask also the $G$-invariance of the dense open subset $\Omega_0$ of $\Omega_1$ (the open set $\Omega_0$ is defined in the condition [H.2]).

We conclude this article with two results (see Section 3.4) relating the SQP meromorphic quotients for the actions of $B$ and $G$ (resp. of $A$ and $G$) when they exist:

1. The existence of a holomorphic map $h : Q_B \to Q_G$ (resp. $Q_A \to Q_G$) between the corresponding quotients.
2. The existence under the hypotheses of the Theorem 1.0.1 (resp. the Theorem 1.0.3) of a $G$-invariant dense Zariski open set $\Omega$ disjoint from the centers of the modifications, such that the corresponding map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$ (resp. $h_\Omega : q_A(\Omega) \to q_G(\Omega)$) is proper.

## 2. Strongly quasi-proper meromorphic quotients

**2.1. Preliminaries.** — For the definition of the topology on the space $\mathcal{C}_n^f(X)$ of finite type $n$-cycles in $X$ and its relationship with the topology of the space $\mathcal{C}_n^{\mathrm{loc}}(X)$ we refer to [4] ch. IV, [2] and [3].

For the convenience of the reader we recall shortly here the definitions of a geometrically f-flat map (f-GF map) and of a strongly quasi-proper map (SQP map) between irreducible complex spaces and we give a short summary on some properties of the SQP maps. For more details on these notions see [2] and [3].